

Theoretical model for mesoscopic-level scale-free self-organization of functional brain networks

Jaroslaw Piersa, Filip Piekniowski, Member, IEEE CIS and Tomasz Schreiber

Abstract—In this paper we provide theoretical and numerical analysis of a geometric *activity flow* network model which is aimed at explaining mathematically the scale-free functional graph self-organization phenomena emerging in complex nervous systems at a mesoscale level. In our model each unit corresponds to a large number of neurons and may be roughly seen as abstracting the functional behavior exhibited by a single voxel under the fMRI imaging. In the course of the dynamics the units exchange portions of *formal charge* which correspond to waves of activity in the underlying microscale neuronal circuit. The geometric model abstracts away the neuronal complexity and is mathematically tractable which allows us to establish explicit results on its ground states and the resulting charge transfer graph modeling functional graph of the network. We show that for a wide choice of parameters and geometrical set-ups the model yields a scale-free functional connectivity with exponent approaching 2 in agreement with previous empirical studies based on fMRI. The level of universality of the presented theory allows us to claim that the model does shed light on mesoscale functional self-organization phenomena of the nervous system, even without resorting to closer details of brain connectivity geometry which often remain unknown. Material presented here significantly extends our previous work where a simplified *mean-field* model in a similar spirit was constructed, ignoring the underlying network geometry.

Index Terms—scale-free, geometric neural network, mesoscale, mesodynamics

I. INTRODUCTION

SCALE-FREE self-organization has attracted considerable amount of attention in statistical mechanics, physics and mathematics. Power laws in degree distributions have been reported in various networks ranging from the World Wide Web [1], science collaboration networks [2], citation networks [3], ecological networks [4], linguistic networks [5], cellular metabolic networks [6], [7] to telephone call network [8], [9] etc. This subject is also tempting for neural network and neuroscience community. There are various reports suggesting performance gains with artificial NN built on power law and small world graphs [10]–[12] and more research is in progress. The studies of biological neural network connectivity of *C. elegans* worm revealed exponential rather than polynomial decay in degree distribution [13], [14], however the worm

has only about 300 neurons total and therefore the self-organization in the sense discussed in this paper is very limited in this case. It would also be surprising to find a power law connectivity on the level of single neurons due to physical limitations - similarly the Internet on the level of hardware is not a scale-free network since there are no physical devices that could connect thousands of computers.

However, recent advances in medical imaging and data processing techniques have provided very strong hints that power law connectivity might constitute an important aspect of self-organization and dynamics of far more complex (mammalian like) nervous systems (in particular neocortex). Whereas we refer the reader to [15], [16] for an extensive review, here we only briefly recapitulate the general picture emerging from this interdisciplinary research. In general, both the structural (anatomic) and functional (based on observed activity profile) networks in human and higher animal brains are found to exhibit highly structured hierarchic architecture of small-world type (small mean path-lengths and high local clustering), see *ibidem* and the references therein, yet not always are these networks scale-free (exhibiting power law in degree distribution). Among various network types studied, depending on the medical imaging and data processing methodology, our interest in this paper is focused on *voxel-based* functional networks arising in fMRI brain observations [17]–[20] where scale-free architecture has been reported, often with exponent approaching 2, both in the rest state and in the course of some task execution. This power law gets exponentially cut off or even breaks down on larger scales, for instance with anatomical regions playing the role of nodes, [21], [22] which is apparently due to the scale difference and to system's physical limitations, see the discussion in [15, p.192]. In fact, it should not be considered surprising that the scale-freeness, whose mathematical definition involves system sizes tending to infinity, at the level of large but finite systems is encountered only in approximate form, moreover it is also very natural that the finite-size corrections arising (for instance the exponential cut-off) become more significant as the system size decreases, for instance due to decreasing resolution, as reported e.g. in [23]. This is a usual situation in statistical physics where the quality of approximation of a complex system's behavior by its mathematical idealization derived in the thermodynamic limit essentially depends on system's size and scale. However, determining the precise range of characteristic spatial and temporal scales as well as of data processing schemes where the scale-free approximation of brain's functional topology is valid, seems to be an important open question upon which many competing opinions exist, see again [15, p.192].

J. Piersa, F. Piekniowski and T. Schreiber are affiliated with the Faculty of Mathematics and Computer Science, Nicolaus Copernicus University, Toruń, Poland. This research has been supported by the Polish Minister of Scientific Research and Higher Education grant N N201 385234 (2008-2010). The work of J. Piersa has also been supported by the European Social Fund and Government of Poland as a part of Integrated Operational Program for Regional Development, ZPORR, Activity 2.6 "Regional Innovation Strategies and Knowledge Transfer" of Kuyavian-Pomeranian Voivodeship in Poland.

Manuscript received ???; revised ???.

The research triggered by these and related empirical findings has often been complemented by computer simulations [24]–[26]. Interestingly, also in some simulation studies based on phenomenological neuron model by E. Izhikevich [27], the presence [25] or absence [28, Chapter 6] of scale-freeness strongly depended on the definition of network’s constituent nodes.

An important step in theoretical understanding of these scale-free self-organization phenomena was made in [29], where it is argued and shown by a numerical study that mesoscopic scale brain functional networks exhibit in many aspect behavior typical for the Ising model at criticality. This is further supported by numeric study in [30] where a further model due to Kuramoto is also considered at criticality.

The empirical findings quoted above have so far lacked a satisfactory dedicated theoretical model, since the traditional ways of constructing scale-free networks [31]–[33] were based on the concept of growth and preferential attachment/duplication - ideas well suited for networks like the Internet, citation graph or even the molecular networks, but not necessarily appropriate for the nervous system. The mean field *spike flow* model introduced in [34] and further studied in [28], [35] was the first mathematically tractable approach to provide such a theoretical background. The model is intended to describe functional networks arising on mesoscopic spatial scales (at the level of fMRI voxels, possibly also on the finer scale of local assemblies of cortical microcolumns, but *not* on the scale of individual neurons) and mesoscopic time scales, long enough to allow for system’s relaxation but short enough to safely assume fixed environment of the dynamics (quenched disorder). It is important to emphasize that on the chosen mesoscale each functional node is composed of thousands of neurons and possibly a number of cortical minicolumns, never of just individual neurons. As we proved mathematically in [35], our model does indeed result in a power law graph with exponent equal 2 (in agreement with many empirical findings as quoted above) based solely on its dynamical features which reflects the charge exchanges/activity flow in a neural network on a medium (meso) scale. In this paper we present an important geometric extension to this *spike flow model*, which goes significantly beyond the mean-field setting and introduces geometric distances between the units thus making the model a lot more feasible for the description of the real nervous system by embedding it in a physical space. We show mathematically and present numeric evidence that also this spatially embedded model does give rise to power law connectivity. In the course of theoretical analysis we also identify finite range distortions marking the validity region of the power law in finite systems. Strikingly, not only is the power law exponent again 2 here, but under certain reasonable regularity conditions it does not depend on parameters of the geometric embedding! These criticality and universality features stand in strong conceptual agreement with [29], [30], although the models considered there pertain to a different spatial scale – an individual state variable in the Ising model is $\{+1, -1\}$ -valued, in Kuramoto’s model it takes values in the unit circle, whereas in our model state variables are unbounded and can be any natural numbers thus

corresponding to more complex entities. Also, the criticality of our model is self-organized (not driven by parameter tuning) as opposed to the cases discussed in [29], [30] where the dynamic criticality is reached by manipulating the control parameters. Moreover, our model evolves under Kawasaki dynamics as opposed to the spin flip Glauber dynamics considered in [29]. Some further discussion of these differences and relationships between our model and [29], [30] will be given at the end of Section IV although we will not go far beyond conceptual comparisons there because these models work on essentially different scales.

The rest of the paper is organized as follows: in section II we describe our basic model which constitutes an important enhancement of the *spike flow* model introduced previously, as being now equipped with geometrical connectivity of a rather general type. We then analyze the ground states and provide a simple description of the asymptotic dynamics in section III. The crucial section IV contains the most important results of the paper related to the emerging scale-free connectivity. In section V we equip the model with additional long range connections which might correspond to long myelinated neuronal fibers connecting remote sites of the cortical surface. Section VII contains a number of numerical results supporting the discussed claims. Finally we conclude the paper and provide references.

II. THE BASIC ISOTROPIC MODEL

The spin glass type model constructed in this section is aimed at abstracting in a mathematically tractable way the essential features determining the mesoscopic level geometry of spiking activity in recurrent neural nets. This was first done in a mean-field setting in [35] where we studied a model built on a complete graph, with all constituent units fully interconnected. Here we go significantly beyond the mean-field analysis and introduce a rather flexible geometric set-up with a considerable amount of degrees of freedom. For definiteness we construct our system on a (unit) sphere $S_{d-1} \subset \mathbb{R}^d$, referred to as the spatial domain of the model in the sequel, and usually taken to be S_2 in our simulations. For intensity parameter $\lambda > 0$ we write $\mathcal{P}_\lambda$ to denote the homogeneous Poisson point process on S_{d-1} with intensity λ , modeling the spatial locations of the units constituting our system. The intensity λ will always be taken large in our simulations, which is reflected by letting $\lambda \rightarrow \infty$ in our theoretical argument. Each unit $x \in \mathcal{P}_\lambda$, sometimes also called a neuron by a convenient abuse of terminology, is endowed with its *state variable* $\sigma_x \in \mathbb{N} := \{0, 1, 2, \dots\}$. In terms of the original neural network the σ_x ’s are to be regarded as an abstraction of *activity levels* of coarse-grained mesoscopic portions of the network at the corresponding spatial locations distributed according to $\mathcal{P}_\lambda$. An alternative interpretation can be also provided under the dynamical system description of the neurodynamics, where σ_x can be regarded, on a mesoscopic time scale, as abstracting the (fraction of) time spent by the system within a behavioral pattern locally manifested at x . For terminological convenience, in the sequel we shall refer to σ_x as to the *charge stored at x* , which makes our developed model

into a *charge flow model*. The *charge conservation* property enjoyed by the stochastic dynamics we construct below, admits a natural motivation in terms of both abstract interpretations of σ_x discussed above, corresponding respectively to *total activity conservation* and the fact that system sojourn times sum up to a constant. To proceed, we consider a *connectivity function* $g : \mathbf{R}_+ \rightarrow [0, 1]$ and for each pair $x, y \in \mathcal{P}_\lambda$ we independently place a connection (edge) between x and y with probability $g(|x - y|)$ and leave these units unconnected with the complementary probability. We write $x \sim y$ to denote the fact that x and y are connected by an edge. Note that the so-constructed connectivity structure of our system corresponds to a classical *random connection model* considered in stochastic geometry [36]. To each edge joining $x \sim y \in \mathcal{P}_\lambda$, denoted below by e_{xy} , we independently assign a standard Gaussian weight $w_{xy} \sim \mathcal{N}(0, 1)$, where, in intuitive terms, a positive w_{xy} should be interpreted as a propensity of x and y to synchronize and thus to exhibit similar activity levels whereas, conversely, a negative w_{xy} indicates an inhibitory interaction between x and y . The energy function of the system (the Hamiltonian) is given by

$$\mathcal{H}(\bar{\sigma}) := \sum_{x \sim y} w_{xy} |\sigma_x - \sigma_y|. \quad (1)$$

The initial configuration of the network is constructed by endowing each unit with a constant initial charge $\alpha \in \mathbf{N}$. Next, the system evolves under a standard Kawasaki-type charge-conserving dynamics. At each step we randomly choose a pair of connected units (σ_x, σ_y) , $x \sim y \in \mathcal{P}_\lambda$ and denote by $\bar{\sigma}^*$ the network configuration resulting from the original configuration $\bar{\sigma}$ by decreasing σ_x by one and increasing σ_y by one, that is to say by *letting a unit charge transfer from σ_x to σ_y* , whenever $\sigma_x > 0$. Next, if $\mathcal{H}(\bar{\sigma}^*) \leq \mathcal{H}(\bar{\sigma})$ we accept $\bar{\sigma}^*$ as the new configuration of the network whereas if $\mathcal{H}(\bar{\sigma}^*) > \mathcal{H}(\bar{\sigma})$ we accept the new configuration $\bar{\sigma}^*$ with probability $\exp(-\beta[\mathcal{H}(\bar{\sigma}^*) - \mathcal{H}(\bar{\sigma})])$, $\beta > 0$, and reject it keeping the original configuration $\bar{\sigma}$ otherwise, with $\beta > 0$ standing for an extra parameter of the dynamics, in the sequel referred to as the inverse temperature conforming to the usual language of statistical mechanics and assumed fixed and large (low temperature) throughout. Observe that the sum $\sum_{x \in \mathcal{P}_\lambda} \sigma_x$ of neuronal charges is preserved by the dynamics and that, in the course of dynamics with some initial configuration $\bar{\sigma}^0$, any other $\bar{\sigma}$ with $\sum_{x \in \mathcal{P}_\lambda} \sigma_x^0 = \sum_{x \in \mathcal{P}_\lambda} \sigma_x$ is eventually reached with positive (although possibly very small) probability. Recall that our default initial configuration choice is $\sigma_x^0 \equiv \alpha$. Consequently, upon standard verification of the usual detailed balance conditions, we readily see that the collection of stationary states of the above dynamics are precisely the distributions

$$\mathbf{P}_n(\bar{\sigma}) = \begin{cases} \frac{\exp(-\beta\mathcal{H}(\bar{\sigma}))}{\sum_{\bar{\sigma}', \sum_x \sigma'_x = n} \exp(-\beta\mathcal{H}(\bar{\sigma}'))}, & \text{if } \sum_x \sigma_x = n, \\ 0, & \text{otherwise} \end{cases} \quad (2)$$

and their convex combinations. In particular, upon forgetting its geometric structure, our model bears formal resemblance to the usual stochastic Boltzmann machines [37] (under Kawasaki dynamics) and inherits the usual intuitive interpreta-

tion of their energy function, with the weights w_{xy} indicating the extent to which the system favors the agreement (for positive w_{xy}) or disagreement (for negative w_{xy}) of the activity levels σ_x and σ_y as already mentioned above. In spite of these formal analogies our model does nevertheless differ from Boltzmann machines in a very essential way due to the unbounded state space $\mathbf{N}$ for σ'_x s. For the network dynamics running during a long time period $[0, T]$ we are now in a position to define the *spike flow graph* to be a directed graph with vertices corresponding to the units σ_x , $x \in \mathcal{P}_\lambda$ and whose edges carry numbers (edge multiplicities) $F_{x \rightarrow y}$ indicating how many times in the course of the dynamics the charge flow occurred from σ_x to σ_y , $x, y \in \mathcal{P}_\lambda$. As we will see, if β is very large, which is always going to be assumed in this paper, after a long enough simulation run the system freezes in some ground state whereupon any further potential flow becomes very rare and consequently the numbers $F_{x \rightarrow y}$ also effectively freeze undergoing virtually no further changes. We say that the dynamics *jams* or *saturates* in this case and we always consider $F_{x \rightarrow y}$ *at saturation* in the sequel, the way of detecting saturation is discussed at the very end of the next Section III, see condition **[Saturation]** there. The in-degree of a unit σ_x is now defined as $d_{\text{in}}(x) := \sum_{y \sim x} F_{y \rightarrow x}$

The crucial question considered in this paper is whether the so-defined spike flow graph is scale-free in that its in-degree distribution follows a power law, that is to say $\mathbf{P}(d_{\text{in}}(x) \approx k) \sim c_{\text{in}} k^{-\gamma_{\text{in}}}$ for a randomly picked node $x \in \mathcal{P}_\lambda$. Below we shall establish a positive answer to this question. It should be noted at this point, as will become clear from our discussion below, that the asymptotic behavior of the corresponding out-degree distribution is the same as that of the in-degrees.

III. WINNER-TAKE-ALL DYNAMICS AND GROUND STATES

It turns out that the long time scale dynamics of our basic model admits a very convenient approximation in terms of a simpler *winner-take-all* dynamics introduced in [35] as soon as λ is large enough, which mathematically corresponds to taking $\lambda \rightarrow \infty$. Our argument here is a version of that given in Section 3 of [35] specialized for our present setting, we refer the reader to [35] for further details. First, we define the *support* S_x of a unit at $x \in \mathcal{P}_\lambda$

$$S_x := - \sum_{y \sim x} w_{xy}. \quad (3)$$

Assume now we run our charge-flow dynamics for some long enough amount of time to get close to equilibrium, whereupon we consider a small number $o(N)$ of units which store the highest charge, considerably higher than the remaining units, and we call these *elite units/neurons* while granting the term *bulk units/neurons* to the remaining units in the system. Since the cardinality of elite is a negligible fraction of N , the formula (1) becomes then

$$\mathcal{H}(\bar{\sigma}) \approx - \sum_{x \in \text{elite}} \sigma_x S_x + \frac{1}{2} \sum_{x, y \in \text{bulk}} w_{xy} |\sigma_j - \sigma_l|.$$

Next, we note that whenever in the course of the network dynamics a charge transfer is proposed from a bulk unit σ_y

to an elite neuron σ_x , $y \sim x$, the resulting energy change is seen to be well approximated by $-S_x$ plus a term due to the interaction between σ_y and other bulk units. In general, we have no control of this term, yet if σ_x is one of the neurons with the highest support in the system, this offending term is very likely to be negligible compared to $-S_x$ thus making the energy change strongly negative and the proposed transfer extremely likely to be accepted. Clearly, the inverse transfer becomes then almost impossible. Consequently, whenever a unit with a very high support enters the elite, it virtually never leaves it; moreover it continuously drains charge from the bulk losing it only to other elite members if at all. Furthermore, should a unit with a small support value happen to enter the elite at the early stages of the dynamics, it will soon leave it having its charge drained by other higher supported units. Thus, after running our dynamics long enough we end up with a picture where the elite consists of units with the highest support. Although the elite neurons do struggle for charge between themselves, they cooperate in draining it from the bulk. Therefore eventually almost no charge will be present in the bulk and hence the Hamiltonian will admit a particularly simple approximation

$$\mathcal{H}(\bar{\sigma}) \approx - \sum_{x \in \text{elite}} \sigma_x S_x \quad (4)$$

and all further updates in the system will only happen due to charge transfers within the elite. Note now that, since the cardinality of the elite is $o(N)$, for each elite neuron its interactions with the remaining elite units are effectively dominated by those with the bulk. Thus, keeping in mind that the inverse temperature β is very large, the dynamics between the elite neurons takes eventually a particularly simple form: a pair $x \sim y$ of connected elite neurons is chosen by random and if the one with smaller support attempts to transfer a unit charge to the one with higher support, the attempt is accepted, otherwise it is rejected. Clearly, this dynamics effectively terminates by storing all available charge in neurons which are not connected to any higher support units, call these *ground units*. Although formally the only ground state of the system is obtained by putting all charge into the unit of the *highest* support, which clearly is a ground neuron, we will slightly abuse the terminology and grant the name of a ground state to each configuration where all charge is stored in ground units – note that this makes a good sense in the $\lambda \rightarrow \infty$ asymptotics where the potential barriers between these *ground states* become impassable for the dynamics. It should be noted that at intermediate stages of the dynamics it may happen that elite members show up with charges whose order is inverse to that determined by the supports rather than consistent with it. This is a metastable artifact due to the fact that if we admitted negative charges here, a twofold sign-flip symmetry would be present in the system in full analogy to usual networks with no external field and such inverse ordering would compete with the standard one on equal rights. This is not the case here though because negative charges are not allowed and therefore such inversely ordered structures are temporarily stable and do not persist in the course of the dynamics.

In view of the above discussion, the highest in-degrees of

the spike-flow graph are observed in elite units enjoying the highest support from the system, and the corresponding charge flows $F_{i \rightarrow j}$ are mainly due to the internal charge transfers within the elite. Thus, our discussion shows that the asymptotic $\lambda \rightarrow \infty$ behavior of the charge flow graph generated by our network model is accurately described by the following simplified *winner-take-all* (WTA) model:

- The system consists of gas of elite units with spatial locations determined by homogeneous Poisson point process $\mathcal{P}_\varrho$ on $\mathbf{S}_{d-1}$ of intensity $1 \ll \varrho \ll \lambda$.
- Each unit located at $x \in \mathcal{P}_\varrho$ has its state variable denoted as before by σ_x .
- Each unit located at $x \in \mathcal{P}_\varrho$ has its *support mark* μ_x chosen uniformly in $[0, 1]$ independently of the locations and remaining supports of units constituting the system. The neuron σ_x is declared to have *higher support* than σ_y , $x, y \in \mathcal{P}_\varrho$ iff $\mu_x > \mu_y$.
- $\alpha \text{card}[\mathcal{P}_\varrho] = \text{Poisson}(\alpha \varsigma(\mathbf{S}_{d-1}))$ units of charge are sequentially introduced into the system, each time according to the following dynamics
 - first, a unit charge is transferred to a randomly chosen neuron σ_{x_0} , $x_0 \in \mathcal{P}_\varrho$,
 - thereupon it starts jumping to subsequent neurons $\sigma_{x_1}, \sigma_{x_2}, \dots$ where x_{l+1} is randomly chosen among y with $x_l \rightsquigarrow y$, that is to say among $y \sim x_l$ with $\mu_y > \mu_{x_l}$. Whenever a jump from σ_{x_l} to $\sigma_{x_{l+1}}$ is performed, the charge transfer counter $\hat{F}_{x_l \rightarrow x_{l+1}}$ gets incremented by one. These charge counters approximate the corresponding charge flow counts in the original model. An additional counter $\hat{F}_{\perp \rightarrow x}$ is also considered storing the number of units initially assigned to σ_x .
 - eventually the unit charge reaches a ground unit and gets frozen there.
- the in-degrees of the elite neurons in the original network, given by $d_{\text{in}}(x) = \sum_{y \sim x} F_{y \rightarrow x}$ are approximated by the numbers $D_x = \hat{F}_{\perp \rightarrow x} + \sum_{y \sim x} \hat{F}_{y \rightarrow x}$ indicating how many charge units have visited σ_x on their way to a ground neuron.

In other words, in this model the charge transfers always occur from a neuron with smaller support to a randomly chosen better supported one to which a connection is available, whence the term *winner-take-all* dynamics. A few words are due to explain the meaning and assignment mechanism of the support marks. Roughly speaking, our intention is to have μ_x such that

$$[\mu_x \text{card}[\mathcal{P}_\varrho]] \quad (5)$$

stand for the number the unit σ_x occupies in the hierarchy of neurons ordered by ascending supports. Assigning i.i.d. uniform marks to units in $\mathcal{P}_\varrho$ is equivalent to making all support orderings equiprobable, which is natural in view of the rotational symmetry of the system, implying equidistribution of support marks, and in view of the fact that the supports S_x and S_y as defined in (3) for two different units σ_x and σ_y can only share w_{xy} (if at all) among their constituent weights and thus are effectively independent given $\mathcal{P}_\varrho$, likewise for higher

but fixed cardinality collections, which asymptotically grants effective independence of the support marks. Clearly, this construction yields an approximation of the original dynamics, which is only statistically valid, but this is sufficient for our needs:

- the winner-take-all rule happens to be violated, with very small probability though,
- the independence of supports given $\mathcal{P}_\varrho$ is only approximate, but this approximation is very good in large ϱ asymptotics,
- the supports S_x and hence μ_x are in general not independent of the spatial configuration $\mathcal{P}_\varrho$ as determining the number of connections at x , and thus μ_x should be made depend on $\mathcal{P}_\varrho$, however this effect can be neglected for our purposes in large ϱ asymptotics as it vanishes already upon local spatial averaging (mesoscopic-level coarse-graining) whereas we mainly consider global average characteristics determining the scale-free geometry of charge flow graphs.

We conclude this section by noting that as a by-product of the discussion above we get a clear criterion determining whether the dynamics of our basic model has reached saturation. This is

[Saturation] The system has reached saturation when all charge is stored at ground units.

IV. SCALE-FREE SELF-ORGANIZATION

The purpose of the present section is to study the degree distribution of the charge flow graph generated by our charge flow model. To this end, we will analyse the behavior of exceedance probabilities of the type

$$G(k) := \mathbf{P}(d_{\text{in}}(x) \geq k) \quad (6)$$

where x is a randomly chosen point in $\mathcal{P}_\lambda$ and where both k and λ are large. Mathematically this corresponds to taking $\lambda \rightarrow \infty$ and $k \rightarrow \infty$ and thus all our theory below is to be understood as statements about the limits as suitable parameters tend to ∞ . The requirement that k be large allows us to restrict our attention to the elite units evolving under the winner-take-all dynamics discussed in Section III. Indeed, the in-degrees of the bulk units are in usually relatively low and thus the bulk neurons can be neglected in our considerations. Consequently, it is enough to study the winner-take-all model with spatial locations distributed according to $\mathcal{P}_\varrho$, as discussed above. To proceed with our analysis, to each point $x \in \mathcal{P}_\varrho$ we ascribe its *visiting intensity* $\Phi(x)$ set to be the expected value of the number D_x of charge units visiting σ_x on their way to a ground neuron, given the configuration $\mathcal{P}_\varrho$. Then it is readily seen by the definition of the WTA dynamics that

$$\Phi(x) = \alpha + \sum_{y \rightsquigarrow x} \frac{\Phi(y)}{\text{card}[N^\uparrow(y)]} \quad (7)$$

where $y \rightsquigarrow x$ stands for $y \sim x$ and $\mu_y < \mu_x$ whereas $N^\uparrow(y) := \{z \in \mathcal{P}_\varrho, y \rightsquigarrow z\}$. The relation (7) allows us to evaluate $\Phi(x)$ recursively starting from the minimal points of the $\rightsquigarrow$ -induced order, that is to say points y for which

$N^\uparrow(y) := \{z \in \mathcal{P}_\varrho, z \rightsquigarrow y\}$ is empty, and proceeding upwards in the support hierarchy towards the maximal elements corresponding to the ground neurons of the system. Writing $\mathbf{1}_{x \rightsquigarrow y}$ for the random variable taking value 1 if $x \rightsquigarrow y$ and 0 otherwise we can rewrite (7) as

$$\Phi(x) = \alpha + \sum_{y \in \mathcal{P}_\varrho} \frac{\mathbf{1}_{y \rightsquigarrow x} \Phi(y)}{\sum_{z \in \mathcal{P}_\varrho} \mathbf{1}_{y \rightsquigarrow z}}$$

and recalling that the random variables $\mathbf{1}_{(\cdot) \rightsquigarrow (\cdot)}$ are all independent by the construction of the random-connection graph, we conclude that the collection of $(\Phi(x))_{x \in \mathcal{P}_\varrho}$ exhibits in large ϱ asymptotics strong self-averaging properties which, together with the spherical symmetry of the system, implies that the value of $\Phi(x)$ effectively only depends on the value of the support mark μ_x , that is to say

$$\Phi(x) \approx \phi(\mu_x) \quad (8)$$

where $\phi(\cdot)$ is some deterministic function. Moreover, for given (x, μ_x) and $y \in \mathcal{P}_\varrho$ with μ_y unknown we have

$$\mathbf{P}(y \rightsquigarrow x) = \mu_x g(|x - y|), \quad \mathbf{P}(x \rightsquigarrow y) = (1 - \mu_x) g(|x - y|) \quad (9)$$

whereas for μ_y known we get

$$\mathbf{P}(y \rightsquigarrow x) = \mathbf{1}_{\mu_y < \mu_x} g(|x - y|). \quad (10)$$

Hence

$$\text{card}[N^\uparrow(x)] \approx \varrho(1 - \mu_x) \int_{\mathbf{S}_{d-1}} g(|x - y|) \varsigma(dy) = \varrho(1 - \mu_x) \gamma \quad (11)$$

with $\gamma := \int_{\mathbf{S}_{d-1}} g(|x - y|) \varsigma(dy)$ not depending on x by isotropy of the system and where $\varsigma(\cdot)$ stands for the spherical surface measure. Note that although (11) is valid for all $\mu_x \in (0, 1)$ in large ϱ asymptotics, for a given ϱ we should have $\varrho(1 - \mu_x) \gg 1$ so that the self-averaging applies. Consequently, putting (7) together with (8,9,10) and (11) we are led to

$$\begin{aligned} \phi(\mu_x) &= \alpha + \varrho \int_{\mathbf{S}_{d-1}} \int_0^{\mu_x} g(|x - y|) \frac{\phi(\mu_y)}{\varrho(1 - \mu_y) \gamma} d\mu_y \varsigma(dy) = \\ &= \alpha + \int_0^{\mu_x} \frac{\phi(y)}{(1 - \mu_y)} d\mu_y \left[\frac{\int_{\mathbf{S}_{d-1}} g(|x - y|) \varsigma(dy)}{\gamma} \right] \end{aligned}$$

for $1 - \mu_x \gg \varrho^{-1}$. Recalling now the definition of γ we end up with

$$\phi(s) = \alpha + \int_0^s \frac{\phi(t)}{1 - t} dt, \quad 1 - s \gg \varrho^{-1}, \quad (12)$$

and thus, upon solving,

$$\phi(s) = \frac{\alpha}{1 - s}, \quad 1 - s \gg \varrho^{-1}. \quad (13)$$

To proceed, we use the fact that the average number of points $x \in \mathcal{P}_\varrho$ with $\mu_x \in [s, s + ds]$ is $\varrho \varsigma(\mathbf{S}_{d-1}) ds$, whereas the overall mean number of points in $\mathcal{P}_\varrho$ is $\varrho \varsigma(\mathbf{S}_{d-1})$, whence the average fraction of points with support marks in $[s, s + ds]$ is simply ds . Moreover, we note that by the self-averaging

property of our system, as $\varrho \rightarrow \infty$ we have $D_x \approx \Phi(x) \approx \phi(\mu_x)$. Therefore, using (13) and recalling (6) we come to

$$G(k) \approx \mathbf{P}(D_x \geq k) \approx \int_{\{s, \phi(s) \geq k\}} ds = \alpha/k. \quad (14)$$

This is the cumulative version of the statement that $\mathbf{P}(d_{\text{in}}(x) \approx k) \propto k^{-2}$ with x standing for the random point in $\mathcal{P}_\varrho$. Thus, we have established the first main theoretical result of this paper.

Theorem 1: The charge flow graph induced by our basic isotropic model is, in large λ and ϱ asymptotics, scale-free with exponent 2, with overwhelming probability. Note that this theorem supports the viewpoint presented in [29], [30] where universality classes of statistical mechanical models at criticality have been advocated as providing possible explanation of crucial properties of brain functional networks. Clearly, there are important differences in mathematical details between our model and the approach presented there, which are due to different spatial scales – the models in [29], [30] refer to simple state variables whereas we are dealing with unbounded state variables corresponding to more complicated mesoscopic units abstracting neuronal ensembles (for instance at the level of fMRI voxels, possibly also on a lower scale of groups of cortical microcolumns) rather than individual neurons. Moreover, the models used in [29], [30] achieve criticality upon adjusting suitable order parameters whereas our model exhibits self-organized criticality as the power law in Theorem 1 does not depend on any order parameters. However, at the conceptual level there is a good agreement. In this context it is worth noting for instance the presence of metastable inverted configurations in the course of the basic model’s dynamics as discussed just above the formal definition of the WTA model in Section III. Likewise, it should be observed that although the Winner-Take-All dynamics itself is very robust, the notion of the winner is not – the system will unfold according to the asymptotic WTA description for overwhelming majority of disorder (weights) configurations, but the support values determining the corresponding support marks and thus the WTA hierarchy are themselves extremely sensitive to the disorder. Moreover, as observed, discussed and presented in figures in Subsection VII-A, the ground units give rise to local *basins of attraction* which exhibit certain level of geometric contiguity and which consist of units whose activities correlate in the course of the dynamics. All these features are again in agreement with critical properties of brain functional networks in focus of [29]. Questions related to this fact are the subject of our ongoing work in progress.

A careful reader might wonder at this point why we did not mention above one further apparently crucial difference between our model and the approach in [29], namely that our model here evolves according to charge-conserving Kawasaki dynamics whereas there the Ising model is let evolve under Glauber spin flip dynamics. The point is that, as can be seen in the proof of Theorem 2 and especially in (18), if we relax the charge conservation the model still exhibits scale-free behavior although with a different exponent. In other words, within reasonable limits the particular choice of the dynamics is not crucial for the scale-freeness property, although clearly the

very nature of our *charge-flow* model requires that at each step two units be chosen, one as the source and the other as the destination, which excludes for instance any kind of Glauber-type dynamics.

We conclude this section by making one further remark. All our results stated in this paper have the form of limit theorems, that is to say statements about limits when suitable parameters tend to ∞ . Although we have some partial knowledge about the speed of this convergence for WTA regime, obtained using measure concentration techniques, see proofs of Corollaries 1 and 2 in [38], and although we are able to estimate certain finite size corrections, see (12) and (13), in general currently we do not know precisely how large λ should be so that our asymptotic results provide a good approximation. However, a very practical answer to this question is given in Subsection VII-A where the system sizes and simulation lengths turned out to be large enough to ensure validity of our theoretical approximation, see Table I.

V. HANDLING LONG RANGE CONNECTIONS

Our basic model admits a natural extension to neatly deal with long range random connections. These connections are assumed to occur along strongly elongated myelinated axons (covered by myelin sheaths) and their impulse propagation time scales are larger than that characteristic for the mesoscopic self-organizing activity of the network considered in our basic model. Consequently, from the viewpoint of the mesoscopic time scale in the focus of our attention, the spike transfers along long myelinated axons do not participate in the self-organizing activity of the network and are locally felt just as a random excitatory noise. Note however that their influence can be important and even crucial for some higher order self-organization on larger time scales, which falls outside the scope of this paper though. In view of the above, the effect of long axonal connections in our mesoscopic setting is modeled by gradual fading or even suppressing the contribution of these spike transfers to dynamic self-organization and regarding them just as a long range propagation mechanism for excitatory noise. In terms of our models it amounts to the following modification to their construction and dynamics, with the WTA version approximating the asymptotic behavior of the original dynamics by argument fully analogous to that given in Section III for our basic set-up:

- We fix a parameter $\delta > 0$ and from each neuron σ_x in the system we emit a mean δ number of *exceptional connections*, modeling the long axons, with target units independently chosen from law $\kappa(|x - y|)\zeta(dy)$, where $\int_{\mathbf{S}_{d-1}} \kappa(|u|)\zeta(du) = 1$. The newly created exceptional connection between two units already connected replaces the previous usual connection. The fact that there is an exceptional connection between units σ_x and σ_y is denoted by $x \Leftrightarrow y$.
- In both the original model and its winner-take-all approximation, whenever a charge transfer is attempted from x to y with $x \Leftrightarrow y$, this attempt is always successful, respectively regardless of the sign of the resulting energy change and regardless of whether $\mu_y > \mu_x$ or not.

Note that the standard way of constructing the exceptional connections with properties required above is to independently place an exceptional connection between each two units σ_x, σ_y

- with probability $\frac{\delta\kappa(|x-y|)}{\lambda\varsigma(\mathbf{S}_{d-1})}$ for $x, y \in \mathcal{P}_\lambda$ in the basic model,
- with probability $\frac{\delta\kappa(|x-y|)}{\varrho\varsigma(\mathbf{S}_{d-1})}$ for $x, y \in \mathcal{P}_\varrho$ in the WTA model.

In both cases, the obtained cardinality of exceptional connections outgoing from an individual unit is precisely Poisson(δ) as required.

Unlike in our basic set-up, here with the exceptional connections present the dynamics in general does not terminate. Thus, to avoid infinite weights on the edges of the charge flow graph, we resort to a technical trick: choose a constant $\epsilon \ll 1$ and, both for the original model and its WTA approximation, each time a charge transfer is attempted, prior to this an additional *survival test* is applied, passed with probability $(1 - \epsilon)$ and failed with the complementary probability ϵ . A charge unit having failed such a test is discarded from the system, whereas a unit having passed continues its evolution. The correct behavior of the system is recovered upon letting $\epsilon \rightarrow 0$. Writing $x \Rightarrow y$ iff either $x \rightsquigarrow y$ or $x \Leftrightarrow y$ and putting $N^\uparrow(y) := \{z \in \mathcal{P}_\varrho, y \Rightarrow z\}$ we see that, in full analogy to (7), here we have

$$\Phi(x) = \alpha + (1 - \epsilon) \sum_{y \Rightarrow x} \frac{\Phi(y)}{\text{card}[N^\uparrow(y)]}. \quad (15)$$

Since the difference of cardinalities $\text{card}[N^\uparrow(y)]$ and $\text{card}[N^\uparrow(x)]$ is of order $O(\delta) = O(1)$ by our construction (recall that δ stays fixed as ϱ varies), their asymptotics are equivalent as long as both these expressions are $\gg \delta$, that is to say as long as $\varrho(1 - \mu_x) \gg \delta$. Moreover, for similar reasons, summing over $y \Rightarrow x$ rather than over $y \rightsquigarrow x$ as in (7) above, makes non-negligible difference only if $\varrho\mu_x = O(\delta)$ (recall that $\varrho\mu_x\gamma$ is the expected cardinality of $N^\downarrow(x)$, cf. (11)). Consequently, as in (12) we get

$$\phi(s) = \alpha + (1 - \epsilon) \int_0^s \frac{\phi(t)}{(1 - t)} dt, \quad \varrho s \gg \delta, \quad \varrho(1 - s) \gg \delta, \quad (16)$$

whence

$$\phi(s) = \frac{C}{(1 - s)^{1-\epsilon}}, \quad \varrho s \gg \delta, \quad \varrho(1 - s) \gg \delta, \quad (17)$$

where the order of the constant C can be readily estimated as follows. We check how many charge units may, on average, visit a minimal point $x \in \mathcal{P}_\varrho$, that is to say a point x with no $y \rightsquigarrow x$, $y \in \mathcal{P}_\varrho$, whence in particular $\mu_x \approx 0$. Clearly, at the initial stage of the dynamics we can expect visits of mean α charge units with their initial assignment at x . Next, each of the $O(\alpha\varrho)$ units present in the system survives on average $1/\epsilon$ jumps (geometric lifetime with mean $1/\epsilon$), at each stage having the probability $O(\delta/\varrho)$ of hitting a vertex connected to x by an exceptional connection, and the probability $O(1/(\gamma\varrho))$ of following this connection. This yields $C \approx \phi(0) = \alpha + O(\alpha\varrho \cdot 1/\epsilon \cdot \delta/\varrho \cdot 1/(\gamma\varrho)) = \alpha(1 + O(\delta/(\epsilon\gamma\varrho)))$. Assuming that $\epsilon\varrho \gg 1$ we end up with $C \approx \alpha$ and thus, as in (14),

$$G(k) \approx \int_{\{s, \phi(s) \geq k\}} ds \approx (\alpha/k)^{1/(1-\epsilon)}. \quad (18)$$

Letting $\varrho \rightarrow \infty$ and $\epsilon \rightarrow 0$ yields therefore

Theorem 2: The charge flow graph induced by the modified model with long axonal delays is, in large λ, ϱ and small ϵ asymptotics, scale-free with exponent 2, with overwhelming probability.

VI. NON-HOMOGENEOUS SET-UP

Consider now a situation where the distribution of neuronal spatial locations is no more isotropic and neither is their connectivity function. We argue below that also in these cases the scale free behavior of the charge flow graph is usually present and the power law exponent is again 2. Our argument is quite sketchy because the considered general scenario includes a plethora of specific models, yet for many of them the following argument is valid in large density asymptotics. For simplicity we do not introduce exceptional connections here and we place ourselves in context of the approximating WTA dynamics, usually available under reasonable regularity conditions but with support marks no more identically distributed and, to the contrary, with their laws strongly depending on spatial locations. Requiring that (5) be asymptotically valid, we see that we keep the property that the fraction of neurons with support marks in $[s, s+ds]$ is ds . Next, we write $\gamma(t, s)$, $t < s$, for the probability that for randomly chosen x with $\mu_x = t$ and y with $\mu_y = s$ we have $x \rightsquigarrow y$ – whereas in the isotropic case we always had $\gamma(t, s) \equiv \gamma/\varsigma(\mathbf{S}_{d-1})$, here this is no more the case and the function $\gamma(s, t)$ may exhibit quite non-trivial behaviors. Repeating the argument leading to (12) we get the following equation, valid under many reasonable collections of regularity conditions:

$$\phi(s) = \alpha + \int_0^s \frac{\phi(t)\gamma(t, s)}{\int_t^1 \gamma(t, u) du} dt = \alpha + \int_0^s \frac{\phi(t)}{\eta(t, s)} dt, \quad (19)$$

where

$$\eta(t, s) := \frac{\int_t^1 \gamma(t, u) du}{\gamma(t, s)}, \quad t < s.$$

In the isotropic case we always had $\eta(t, s) \equiv 1 - t$, which is no more valid in the general non-homogeneous situation. A simple analysis of the integral equation (19) shows that the behavior of ϕ is usually determined by the properties of η , and thus of γ , in the vicinity of $(1, 1)$. Further, the intuitively natural situation is that the limit $\lim_{s, t \rightarrow 1} \gamma(s, t)$ exists and is positive. We have then $\eta(t, s) \approx 1 - t$ for t close to 1, which does again yield $G(k) \propto k^{-1}$ and hence the charge flow graph is scale-free with exponent 2. Clearly, in general the equation (19) does also constitute a hint how to construct a 'anomalous' system where the scale free property of the charge flow graph fails. However, we believe that these are exceptional situations and we are not aware of natural systems where such anomalies would occur.

VII. NUMERICAL SIMULATIONS

A. Geometric model

In this subsection we present results of numerical simulations for the geometric spike flow model. We work with the basic isotropic set-up introduced in Section II. Neurons

were randomly placed on two-dimensional sphere with radius varying up to 20. The expected density is 10 nodes per square unit of surface, that gives up to 50000 neurons. The synaptic connectivity function $g(\cdot)$, as considered in Section II, is defined here as

$$g(r) = \begin{cases} 1 & r \in [0, 1) \\ \frac{1}{r^\alpha} & r \geq 1 \end{cases} \quad (20)$$

where alpha is fixed at value $\alpha = 2.5$. In particular, this ensures that the network is connected with overwhelming probability. The power-law form of the connectivity function was chosen motivated by qualitative empirical reports in [17]. It should be recalled at this point that although the real-world form of the connectivity function is still a subject of ongoing discussion and may depend on the particular anatomical region, see *ibidem* and [15], our theory does not depend on a particular shape of $g(\cdot)$. The value of inverse temperature β has been set to 1000, which results in rejecting vast majority of changes leading to higher energy states.

The resulting final network state output by simulation included less than 400 units (out of 50000, that is about 0.8 per cent) whose charge was not completely depleted and which are therefore natural candidates to be considered as the ground units of the network, see Section III. Recall that in a fully connected network there is precisely one such ground unit, a single node storing all charge present in the network. In our present geometric situation though there are usually several ground units, enjoying exceptionally high supports (3). Formally to determine when to stop the simulation we should use the criterion [**Saturation**] at the end of Section III, however for practical reasons we have fixed a large enough number of iterations instead (up to 10^9), which allowed us to avoid the costly run-time checks and gave very good convergence to saturation, see the data below.

The cumulative log-log plot of spike flow graph in-degrees vs their frequencies, produced by simulation, is depicted in fig. 2(a). We observe dramatic breakdown of occurrence frequency for high in-degree nodes. This behavior seems explicable when we take into account that high in-degree units are likely not to be connected, which makes the analysis leading to our crucial equation (13) fail to work for the highest support elite – effect discussed in the derivation of (13) and made explicit by validity bounds imposed on the support variable s there. In the middle part of the plot strongly marked linear dependency can be observed. Note that in terms of the original data this reflects a power law with exponent given by the slope of the straight line in the log-log plot. Due to the afore-mentioned abnormal behavior of the plot for highest support elite, a naive least square estimate for this slope, attaining values $a \simeq -1.2964$, is dominated by outlying high in-degree values and also distorted by the statistics of low bulk neurons of supports too low to guarantee WTA-type evolution. However, as discussed in detail in the theoretical part of this paper, we are interested in the behavior of neurons exhibiting average characteristics and thus we reject the highest elite and lowest bulk outliers and only concentrate on the statistics of the clearly pronounced straight line in the middle of the plot. In our case, due to finite simulation size, quite modest compared to real-world neural

system sizes, this corresponds to rejecting about 40 per cent of the data with highest flow values, whereupon LS returned slope parameter $a \simeq -1.063$ which is a good approximation of the theoretical value -1 for the slope of cumulative log-log plot. The cutoff range has been chosen ‘manually’ to get the best linear fit. Generally, it should be emphasized that the greater network size is, the smaller cutoff is needed and better approximation is returned by the simulation.

Interesting structures, emerging during simulation, are highest elite charge flow graphs centered at individual ground elite nodes, i.e. sets of edges, through which large amount of charge flows towards a ground elite unit and nodes which transfer their charges via those edges. It turns out that the entire flow graph centered at a high-support unit usually covers nearly the whole network (see fig. 4(a)). This confirms predictions about the cooperation of elite in draining charge from the bulk units. Therefore, to get an interesting picture we should impose a size limit for a single elite node charge flow graph and only consider edges of highest transfer counts. Having applied these for an individual graph we obtain an „attraction basin” of an elite node (figures 4(b), 4(c)). Basins tend to repel one another, but they are not completely separated because charge flow from bulk neurons branches out among elite units.

Fraction of accepted charge transfers can vary depending on network size and iteration number, yet always rapidly decreases in the course of simulation, marking the convergence to equilibrium/saturation, and hardly ever exceeds two per cent at later stages of network’s evolution. Figure 3(a) depicts these fractions collected over simulation periods (one per cent of consecutive iterations) vs period number, this value rapidly approaches zero (in fact, interestingly enough, we observed it decays as a power function). Fluctuations observable on 3(b) seem to originate from local metastable states visited by the system in the course of the dynamics (see Section III) as well as to zero-to-zero connections removal from the network (which speeds up computation while not affecting the WTA dynamics).

Due to high inverse temperature, number of transfers resulting in energy increase is about few hundreds overall, which is 0.01–0.1% of all accepted changes and thus can be considered negligible.

B. Mean field model with long range connections

In this section we deal with long range connections, whose theory has been given in Section V above, here restricting our simulation to mean field set-up (connectivity function $g \equiv 1$) which was necessary at this stage of our work due to lack of computational power to simulate significantly larger networks as needed to provide statistically relevant outcomes for set-ups where non-trivial connectivity function g and long range connections co-exist (we believe the order of hundreds of thousands would be required). The detailed numerical results for the mean field model in its most basic form are available in [28], [35]. The instances of the mean field model are much smaller than those presented in previous subsection due to all-to-all connectivity and memory limitations (5000 vertices is a reasonable limit for such simulations), yet this is already

Neurons	Connections	Iterations	Slope	Geometry	Units with charge
9k	580k	70M	-1.062	S^2	0.0065
9k	560k	100M	-1.060	S^2	0.01
11k	700k	100M	-1.100	S^2	0.009
12k	800k	100M	-1.118	S^2	0.008
12k	800k	150M	-1.100	S^2	0.009
12k	800k	150M	-1.053	S^2	0.0087
13k	960k	150M	-1.066	$(0..d)^3$	0.008
19k	1.3M	150M	-1.082	S^2	0.011
21k	1.7M	200M	-1.170	$(0..d)^3$	0.008
21k	1.7M	200M	-1.129	$(0..d)^3$	0.01
32k	2.1M	300M	-1.102	S^2	0.0076
33k	2.7M	200M	-1.096	$(0..d)^3$	0.019
40k	3.4M	500M	-1.066	$(0..d)^3$	0.0065
45k	3.0M	500M	-1.103	S^2	0.0082
48k	4.2M	800M	-1.063	$(0..d)^3$	0.0056
48k	4.2M	600M	-1.143	$(0..d)^3$	0.019
50k	3.4M	800M	-1.080	S^2	0.016
50k	3.5M	800M	-1.058	S^2	0.0078
58k	4M	900M	-1.091	S^2	0.0082
58k	5M	1G	-1.081	$(0..d)^3$	0.006

TABLE I: Results of simulation. Table columns include number of neurons, number of connections, number of iterations, approximated slope value, network geometry (S^2 is a sphere, $(0..d)^3$ is three-dimensional cube) and fraction of nodes storing all network charge.

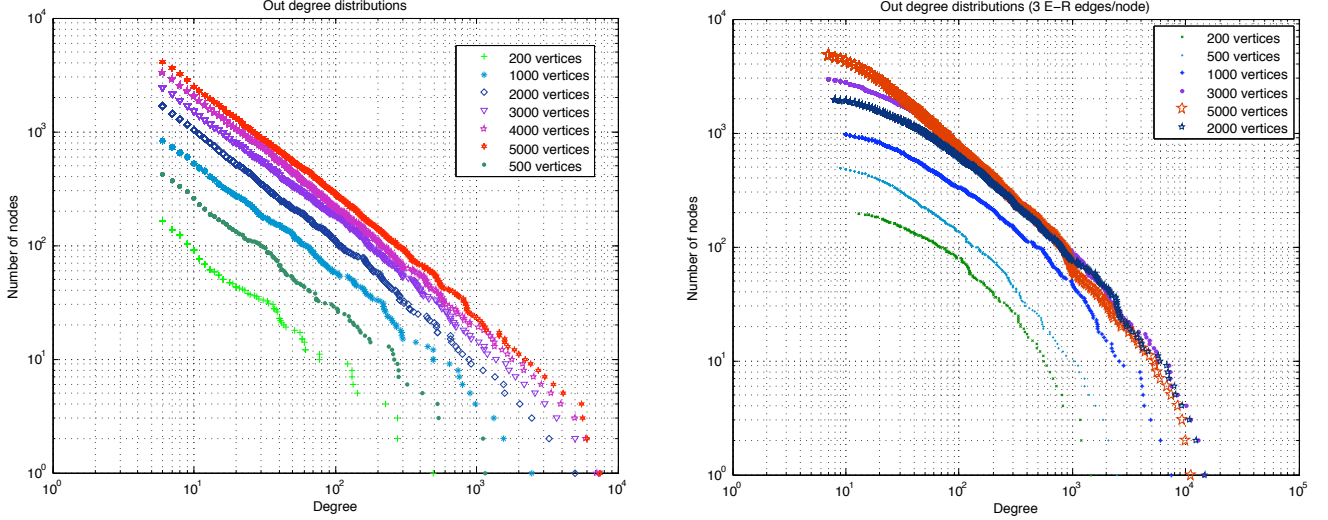
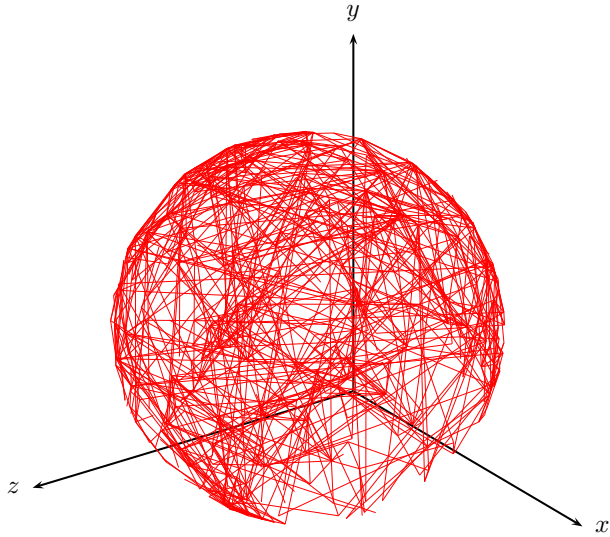


Fig. 5: Degree distributions (complementary CDF, log-log plots) of systems of various sizes plotted together. The left figure shows the original mean field spike flow model, where the power law is well pronounced for a variety of sizes. The right one shows the model equipped with Erdős-Rényi fraction of approximately 3 edges/node. Note that the plot curvature decreases as the number of vertices gets increased. While small systems clearly reveal light tailed decay in vertex degrees, the plot for a system of 5000 vertices (red) is close to linear with the required slope 1 and thus resembles the predicted power law dependency.

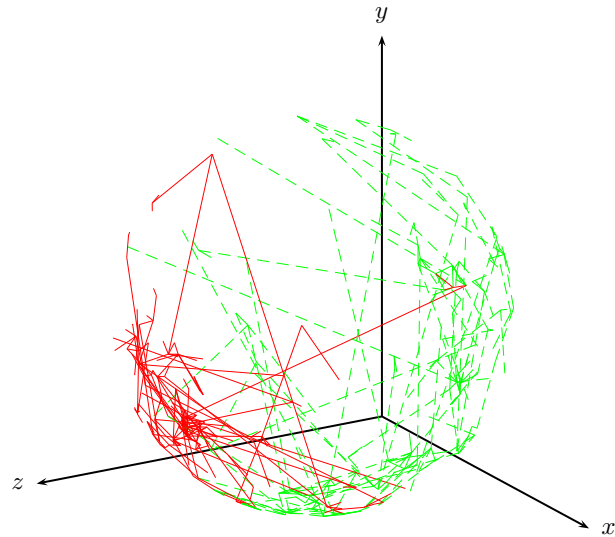
enough to confirm our theoretical predictions because the convergence of network's asymptotic characteristics is much faster here due to the fact, that with more degrees of freedom and all to all connectivity the ground states are easier to find.

Below, we present the degree histograms of the mean field model equipped with random long range connections as discussed in Section V. These connections can be regarded as an Erdős-Rényi random graph imposed on the original all-to-all connectivity structure, along whose edges the charge transfers always get accepted (these *exceptional* edges always conduct charge, neglecting the energy factor). Since in such

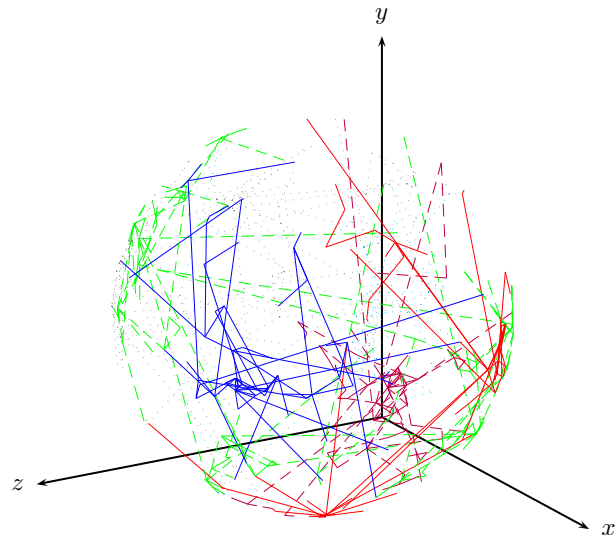
a case the simulation might not converge to a ground state (the charge might cycle forever, distorting the structure of the resulting charge flow graph), the amount of charge was slightly reduced at every time step (each charge unit had a finite geometric lifetime) - this trick as mentioned in Section V ensures that the system does not over-saturate, while preserving all essential features of the dynamics. The figure 4 shows complementary cumulative distributions (CCFD) of systems of three sizes (500, 1000 and 5000 units) under various densities of the exceptional connections. The power laws are fairly well preserved even if the exceptional edges



(a) Charge flow graph covering the whole network.



(b) Two graphs limited only to 200 most important edges per graph.



(c) Six graphs limited only to 100 most important edges per graph.

Fig. 1: Charge flow graphs.

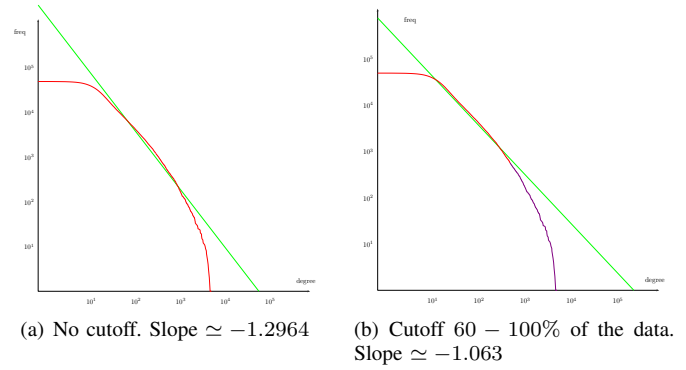


Fig. 2: Plot of degree distribution function and an approximating line.

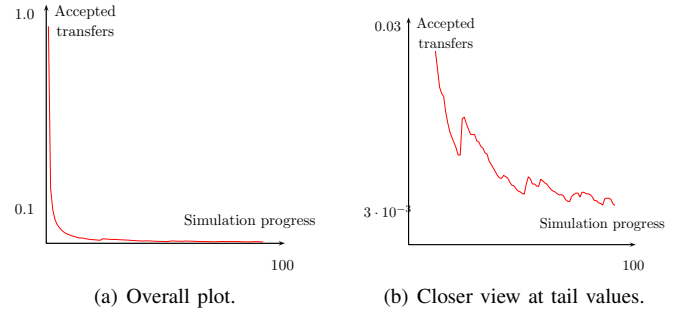


Fig. 3: Fraction of accepted charge transfers in during simulation progress.

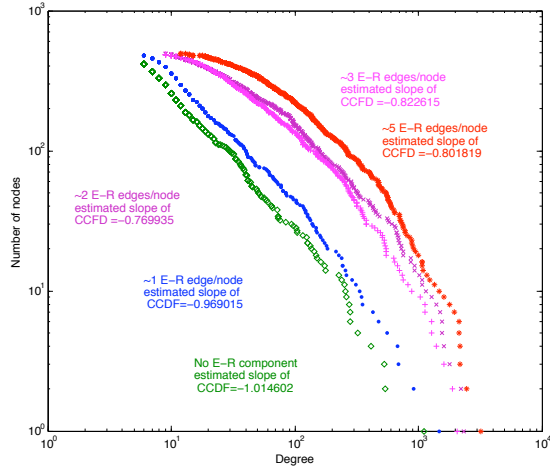
form a giant component (there is more than one edge per node) and are nearly intact for sparser components (below the giant component threshold). At some density the power law breaks down because the simulated system's size is not high enough for the number of exceptional connections to remain negligible. As expected, bigger systems can accept more dense random components and still display power law connectivity as seen in figure 5. More details of the presented numeric results can be found in [39].

As already mentioned above, we haven't run a combined simulation in which both geometry and exceptional connections are present. The point is that we expect that, since both these factors produce strong finite size effects (boundary distortions to the range of node degrees obeying the power law), the required size of simulated system that could reveal the scale-free connectivity in a statistically significant way might be very large.

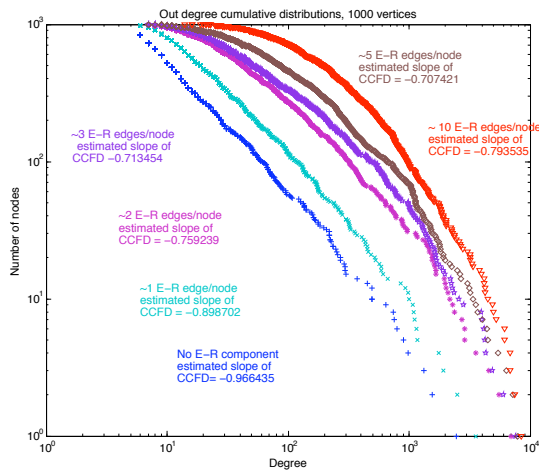
In particular this could be the reason why certain functional networks inherit some structural properties usually found in scale-free networks [24], but lack a well pronounced power law dependence in the degree distribution.

VIII. CONCLUSION

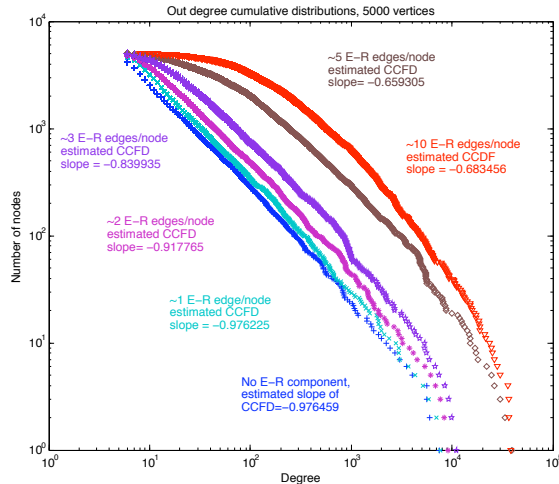
In this paper we have introduced a geometrically-embedded spin glass type theoretical model for mesoscopic scale brain functional networks, essentially extending the mean-field set-up of [35]. We have proved mathematically and verified numerically that, in large system size asymptotics, our model



(a) Results from simulation of 500 vertices.



(b) Results from simulation of 1000 vertices.



(c) Results from simulation of 5000 vertices.

Fig. 4: Sample (out) degree distributions (complementary CDF, log-log plots) of mean field charge flow systems with various density of Erdős-Rényi fractions. Clearly for the smaller system the presence of Erdős-Rényi fraction with approximately 2 edges/node already seriously disturbs the power law. The bigger system maintains scale-free properties until the density of E-R fraction reaches approximately 5 edges/node.

results in scale-free functional networks in agreement with voxel-level fMRI-based empirical findings [17]–[20]. The corresponding power law exponent is 2, which is equal to or not far from the exponents reported there. Moreover, we also have shown that this exponent does not depend on particular choice of the underlying structural connectivity function determining the embedding and even remains mostly invariant under additional modifications of model's geometry and dynamics as discussed in Sections V, VI and Subsection VII-B. These features are indicative of self-organized criticality and universality, in conceptual agreement with [29], [30], although their considerations involve essentially different scale of network units. One further idea in common with [29], [30] is that whereas the dynamics underlying the real-world brain functional networks is extremely intricate and complicated, its essential large scale emergent features can be described by a rather simply formulated mathematical model at criticality (a model in the same *universality class* in the language of statistical physics). In our theoretical study we have also determined validity bounds of the power law connectivity statistics in finite size systems. These bounds are felt in our simulations and result in cut-off power laws when the number of units is limited, which is reminiscent of the cut-off phenomena reported in [21]–[23]. It should be emphasized at this point that the universality claim we make refers only to the presence of scale-freeness and the corresponding exponent, it is clear other important asymptotic features of the network can and often do essentially depend on the underlying connectivity function. A number of further crucial properties have been reported empirically for brain functional networks [15] and it is the subject of our further research in progress to verify these properties and their degree of universality for our model with various geometric embeddings.

ACKNOWLEDGEMENTS

The authors gratefully acknowledge the support from the Polish Ministry of Scientific Research and Higher Education grant N N201 385234 (2008-2010). The work of J. Piersa has also been supported by the European Social Fund and Government of Poland as a part of Integrated Operational Program for Regional Development, ZPORR, Activity 2.6 "Regional Innovation Strategies and Knowledge Transfer" of Kuyavian-Pomeranian Voivodeship in Poland.

REFERENCES

- [1] R. Albert, H. Jeong, and A.-L. Barabási, "Diameter of the world-wide web," *Science*, vol. 401, pp. 130–131, September 1999.
- [2] A.-L. Barabási, H. Jeong, Z. Néda, E. Ravasz, A. Schubert, and T. Vicsek, "Evolution of the social network of scientific collaborations," *Physica A*, vol. 311, no. 4, pp. 590–614, 2002.
- [3] S. Redner, "How popular is your paper? an empirical study of the citation distribution," *European Physical Journal B*, vol. 4, no. 2, pp. 131–134, 1998.
- [4] J. M. Montoya and R. V. Solé, "Small world patterns in food webs," *Journal of Theoretical Biology*, vol. 214, no. 3, pp. 405–412, February 7 2002.
- [5] R. F. i Cancho and R. V. Solé, "The small-world of human language," *Proceedings of the Royal Society of London B*, vol. 268, no. 1482, pp. 2261–2265, November 7 2001.

- [6] U. S. Bhalla and R. Iyengar, "Emergent properties of networks of biological signaling pathways," *Science*, vol. 283, no. 5400, pp. 381–387, January 1999. [Online]. Available: <http://dx.doi.org/10.1126/science.283.5400.381>
- [7] H. Jeong, B. Tombor, R. Albert, Z. N. Oltvai, and A.-L. Barabási, "The large-scale organization of metabolic networks," *Nature*, vol. 407, no. 6804, pp. 651–653, October 5 2000.
- [8] J. Abello, A. Buchsbaum, and J. Westbrook, "A functional approach to external graph algorithms," *ESA '98: Proceedings of the 6th Annual European Symposium on Algorithms*, pp. 332–343, 1998. [Online]. Available: http://dx.doi.org/10.1007/3-540-68530-8_28
- [9] W. Aiello, F. Chung, and L. Lu, "A random graph model for massive graphs," in *STOC '00: Proceedings of the thirty-second annual ACM symposium on Theory of computing*. New York, NY, USA: ACM, 2000, pp. 171–180.
- [10] J. I. Perotti, F. A. Tamarit, and S. A. Cannas, "A scale-free neural network for modelling neurogenesis," *Physica A Statistical Mechanics and its Applications*, vol. 371, pp. 71–75, Nov. 2006.
- [11] D. Stauffer, A. Aharony, L. da Fontoura Costa, and J. Adler, "Efficient hopfield pattern recognition on a scale-free neural network," *The european physical journal B*, no. 32, pp. 395–399, 2003.
- [12] P. N. McGraw and M. Menzinger, "Topology and computational performance of attractor neural networks," *Physical Review E*, vol. 68, p. 047102, 2003. [Online]. Available: <http://www.citebase.org/abstract?id=oai:arXiv.org:cond-mat/0304021>
- [13] L. A. Amaral, A. Scala, M. Barthelemy, and H. E. Stanley, "Classes of small-world networks," *Proc Natl Acad Sci U S A*, vol. 97, no. 21, pp. 11 149–11 152, October 2000. [Online]. Available: <http://dx.doi.org/10.1073/pnas.200327197>
- [14] C. Koch and G. Laurent, "Complexity and the Nervous System," *Science*, vol. 284, no. 5411, pp. 96–98, 1999. [Online]. Available: <http://www.sciencemag.org/cgi/content/abstract/284/5411/96>
- [15] E. Bullmore and O. Sporns, "Complex brain networks: graph theoretical analysis of structural and functional systems," *Nature Reviews Neuroscience*, vol. 10, pp. 186–198, 2009.
- [16] W. J. Freeman, "Scale-free neocortical dynamics," *Scholarpedia*, vol. 2, no. 2, p. 1357, 2007. [Online]. Available: http://www.scholarpedia.org/article/Scale-free_neocortical_dynamics
- [17] V. M. Eguíluz, D. R. Chialvo, G. A. Cecchi, M. Baliki, and A. V. Apkarian, "Scale-free brain functional networks," *Phys Rev Lett*, vol. 94, no. 1, January 2005. [Online]. Available: <http://view.ncbi.nlm.nih.gov/pubmed/15698136>
- [18] R. Salvador, J. Suckling, M. R. Coleman, J. D. Pickard, D. Menon, and E. Bullmore, "Neurophysiological architecture of functional magnetic resonance images of human brain," *Cerebral Cortex*, vol. 15, pp. 1332–1342, 2005.
- [19] G. A. Cecchi, A. R. Rao, M. V. Centeno, M. Baliki, A. V. Apkarian, and D. R. Chialvo, "Identifying directed links in large scale functional networks: application to brain fmri," *BMC Cell Biology*, vol. 8, Suppl. 1, p. S5, 2007.
- [20] M. P. Van den Heuvel, C. J. Stam, M. Boersma, and H. E. Hulshoff Pol, "Small-world and scale-free organization of voxel-based resting-state functional connectivity in the human brain," *Neuroimage*, vol. 43, pp. 528–539, 2008.
- [21] S. Achard, R. Salvador, B. Whitcher, J. Suckling, and E. Bullmore, "A resilient, low-frequency, small-world human brain functional network with highly connected association cortical hubs," *J. Neurosci.*, vol. 26, no. 1, pp. 63–72, January 2006. [Online]. Available: <http://dx.doi.org/10.1523/JNEUROSCI.3874>
- [22] G. Gong, Y. He, L. Concha, C. Lebel, D. W. Gross, A. C. Evans, and C. Beaulieu, "Mapping anatomical connectivity patterns of human cerebral cortex using in vivo diffusion tensor imaging tractography," *Cerebral Cortex*, vol. 19, pp. 524–536, 2009.
- [23] S. Hayasaka and P. J. Laurienti, "Degree distributions in mesoscopic and macroscopic functional brain networks," 2009. [Online]. Available: <http://arxiv.org/abs/0903.4168>
- [24] M. Kaiser, R. Martin, P. Andras, and M. P. Young, "Simulation of robustness against lesions of cortical networks," *European Journal of Neuroscience*, vol. 25, p. 3185, 2007. [Online]. Available: [doi:10.1111/j.1460-9568.2007.05574.x](http://dx.doi.org/10.1111/j.1460-9568.2007.05574.x)
- [25] F. Piękniewski, "Emergence of scale-free graphs in dynamical spiking neural networks," in *Proc. IEEE International Joint Conference on Neural Networks*. Orlando, Florida, USA: IEEE Press, August 2007, pp. 755–759. [Online]. Available: http://ieeexplore.ieee.org/xpl/freeabs_all.jsp?arnumber=4371052
- [26] R. Galán, "On how network architecture determines the dominant patterns of spontaneous neural activity," *PLoS ONE*, vol. 3, p. e2148, 2008.
- [27] E. M. Izhikevich, "Simple model of spiking neurons," *IEEE Transactions on Neural Networks*, no. 14, pp. 1569–1572, 2003. [Online]. Available: <http://www.nsi.edu/users/izhikevich/publications/spikes.pdf>
- [28] F. Piękniewski, "Spontaneous scale-free structures in spike flow graphs for recurrent neural networks," Ph.D. dissertation, Warsaw University, Warsaw, Poland, 2008. [Online]. Available: <http://www.mat.umk.pl/~philip/dissertation.pdf>
- [29] D. Fraiman, P. Balenzuela, J. Foss, and D. R. Chialvo, "Ising-like dynamics in large-scale functional brain networks," *Physical Review E*, vol. 79, p. 061922, 2009.
- [30] M. G. Kitzbichler, M. L. Smith, S. R. Christensen, and E. Bullmore, "Broadband criticality of human brain network synchronization," *PLoS Computational Biology*, vol. 5, p. e1000314, 2009.
- [31] R. Albert and A.-L. Barabási, "Statistical mechanics of complex networks," *Reviews of modern physics*, no. 74, pp. 47–97, January 2002. [Online]. Available: <http://arxiv.org/abs/cond-mat/0106096>
- [32] M. Newman, A.-L. Barabási, and D. J. Watts, *The Structure and Dynamics of Networks: (Princeton Studies in Complexity)*. Princeton, NJ, USA: Princeton University Press, 2006. [Online]. Available: <http://press.princeton.edu/titles/8114.html>
- [33] F. Chung and L. Lu, *Complex Graphs and Networks (Cbms Regional Conference Series in Mathematics)*. Boston, MA, USA: American Mathematical Society, 2006.
- [34] F. Piękniewski and T. Schreiber, "Emergence of scale-free spike flow graphs in recurrent neural networks," in *Proc. IEEE Symposium Series in Computational Intelligence - Foundations of Computational Intelligence*, Honolulu, Hawaii USA, April 2007, pp. 357–362. [Online]. Available: <http://doi.ieeecomputersociety.org/10.1109/FOCI.2007.371496>
- [35] —, "Spontaneous scale-free structure of spike flow graphs in recurrent neural networks," *Neural Networks*, vol. 21, no. 10, pp. 1530 – 1536, 2008. [Online]. Available: <http://dx.doi.org/10.1016/j.neunet.2008.06.011>
- [36] R. Meester and R. Roy, *Continuum percolation*, ser. Cambridge Tracts in Mathematics. Cambridge: Cambridge University Press, 1996, vol. 119.
- [37] E. Aarts and J. Korst, *Simulated annealing and Boltzmann machines: a stochastic approach to combinatorial optimization and neural computing*. New York, NY, USA: John Wiley & Sons, Inc., 1989.
- [38] T. Schreiber, "Spectra of winner-take-all stochastic neural networks," 2008. [Online]. Available: <http://www.citebase.org/abstract?id=oai:arXiv.org:0810.3193>
- [39] F. Piękniewski, "Robustness of power laws in degree distributions for spiking neural networks," in *Proc. IEEE 2009 International Joint Conference on Neural Networks (IJCNN)*. Atlanta, Georgia, USA: IEEE Press, June 2009, pp. 2541–2546.