Spectra of the Spike Flow Graphs of Recurrent Neural Networks

Filip Piękniewski

Faculty of Mathematics and Computer Science, Nicolaus Copernicus Univeristy, Toruń, Poland

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Motivation - correlation networks in fMRI

- Power-law connectivity appears in many real world networks
- Some experiments show, that power-law connectivity emerges in brains on the level of a *correlation network* between centers of activity
- These centers are of fMRI resolution (mm) and are composed of thousands of neurons firing together [Sporns *et al.* , 2004, Eguíluz *et al.* , 2005]
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Theoretical approach - the spike flow model

- The spike flow model (described in detail on the next slide) is a vastly simplified theoretical model that tries to catch some important features of neurodynamics
- Its core idea is that there is a graph filled with units of activity (spikes) that can be exchanged between network's nodes
- Unlike traditional neurons, the nodes in the spike flow model can keep their activity for later, essentially remembering their state
- Such a property can only be found in recurrently connected groups of neurons, which can store their activity via loopback excitation

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The spike flow model

- The model consists of n units $\sigma_i \in \mathbb{N}$, $i = 1 \ldots n$ wired all-to-all with symmetric weights taken from Gaussian distribution
- The energy function is of the form:

$$\mathbb{E}(\overline{\sigma}) := \frac{1}{2} \sum_{i \neq j} w_{ij} |\sigma_i - \sigma_j| \qquad (1)$$

- The dynamics is as follows:
  - We choose randomly two units σ_i, σ_j and check whether $\sigma_i > 0$
  - We check whether a transfer of a charge unit from σ_i to σ_j ($\sigma_i := \sigma_i - 1$, $\sigma_j := \sigma_j + 1$) decreases the energy
  - If so, we accept the change. Otherwise we might accept the change with probability $e^{-\beta \Delta E}$ conforming with the usual stochastic dynamic principles.

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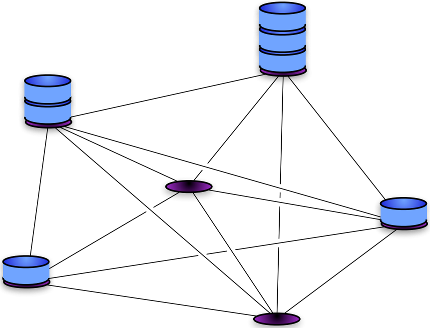
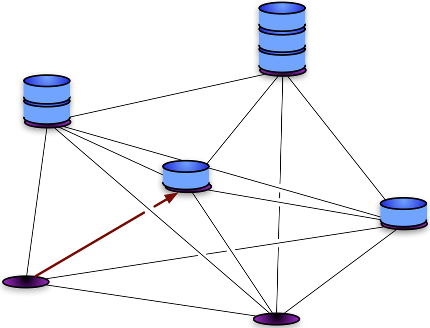


Figure: Schematic description of the *spike flow model*. The nodes contain some amount of tokens (units of charge), which can be exchanged under stochastic dynamics. An event of charge exchange can be seen as a spike (or possibly whole cascade of spikes) whence the name - *spike flow model*.



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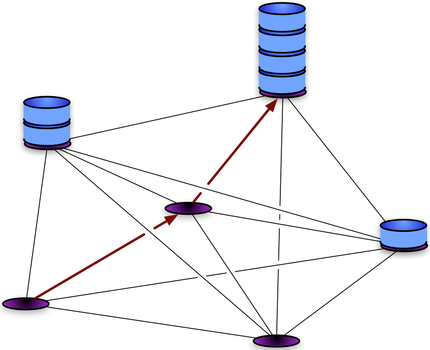

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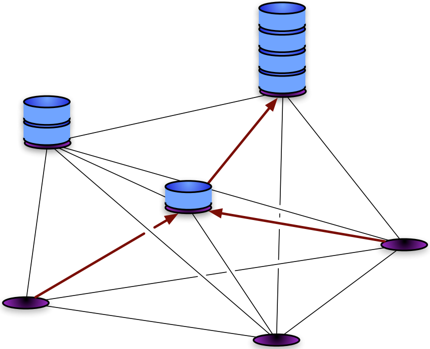

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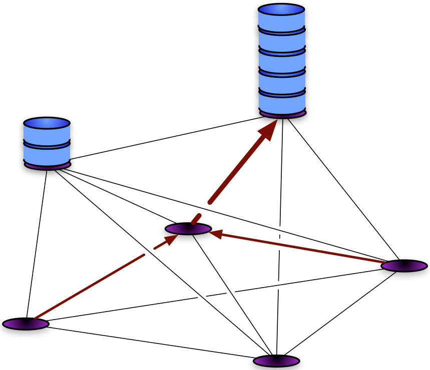

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Properties of the spike flow model

In [Piękniewski & Schreiber, 2008] a number of results related to the *spike flow model* have been established:

- In contrast to a seemingly complex dynamics, with high probability there is a unique ground state of the system, in which all the charge is gathered in a unit that maximizes

$$S_i := -\sum_{j \neq i} w_{ij}. \tag{2}$$

  referred to as *support* in the sequel. The proof goes by a mixture of rigorous and semi rigorous calculations and has a rather asymptotic character, but is in full agreement with numerical simulations for systems containing between a couple hundreds to a couple of thousands of nodes.

Properties of the spike flow model

- The system's behavior eventually admits a particularly simple approximation in terms of a kind of winner-take-all dynamics: almost all transfers converge to units of maximal support (referred to as the *elite* , while the others referred to as the *bulk*), which then compete in draining charge from each other. Ultimately the unit of maximal support gathers all of the charge and the system freezes in a ground state.
- The node degree distribution (where by degree we mean the sum of counters of edges adjacent to a given node) obeys a power law with exponent $\gamma = 2$. The proof is based on the elite/bulk approximation and properties of ordering sequences. Again there is a strong agreement with numerical results.

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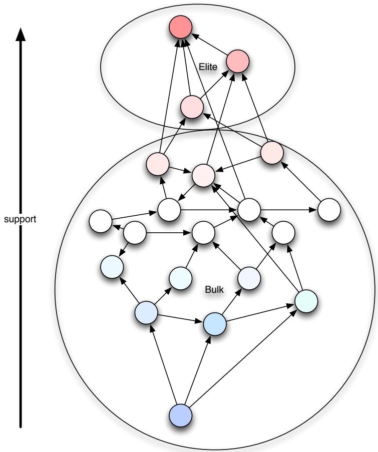
The Spike flow graph

The schematic presentation of the spike flow graph. Within the elite the charge flows (almost surely) in the direction of increasing support. In the bulk the individual weights between the units are more significant and the transfers are less organized.



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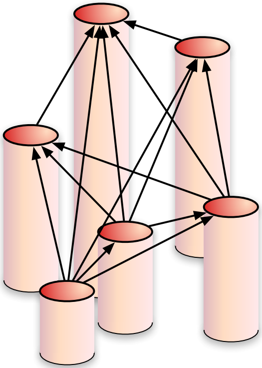
Asymptotically the model becomes particularly simple - the supports are Gaussian sums and the potential with high probability climbs up in the support hierarchy. Eventually all charge ends up in a single unit of highest support.



How to compare networks? The Internet lesson.

- Are the empirical networks and the spike flow graphs really similar?
- The Internet lesson [Li *et al.* , 2005, Willinger *et al.* , 2009] teaches us that relying solely on the degree distribution can be misleading.
- [Achard *et al.* , 2006] studied rest state functional networks and found features characteristic for power law networks but they failed to observe the power law distribution itself
- We claim, that its worth to study spectral properties of networks and compare spectral regimes to estimate networks similarities. In particular we argue that the spike flow graphs have certain asymptotical fingerprints that can be found in empirical networks.

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Spectral characteristics

- In general spectrum (set of eigenvalues of the adjacency matrix of a graph) contains a lot of information about the graphs connectivity, though most of that information might be hard to read and non local in terms of the graph itself.
- Similar graphs have similar spectral properties (expansion properties etc. see [Chung, 1997]), however there can be many graphs with exactly the same spectrum
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Theoretical results regarding spectra of the SFG

- The theoretical foundation for spectral analysis of the Spike Flow Graphs has been established in [Schreiber, 2008] (paper in review),
- The results were established by investigating the spectrum of appropriate Hilbert-Schmidt type operators associated to the random evolution in the asymptotic regime, valid for the simplified WTA *spike flow graph* truncated at both ends by some δ_1 and δ_2 i.e. the nodes of degree less than δ_1 and more than δ_2 are removed from the graph.
- The main theorem of [Schreiber, 2008] implies that when sorted descending, the k-th eigenvalue behaves like $\frac{C}{k^2}$ for some constant C

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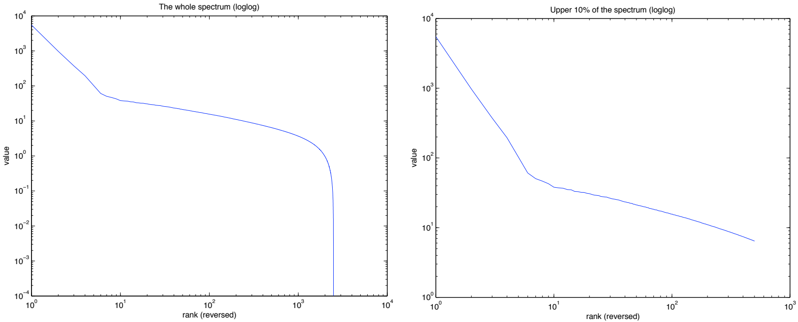

Figure: An example spectrum of a graph consisting of 5000 vertices. The eigenvalues were sorted descending and plotted on a log-log plot. The left figure shows the whole spectrum, the right one only 10% largest eigenvalues. Clearly, the left linear part of the plot behaves like $y = -2x$ which implies that k-th eigenvalue is proportional to k^{-2}. This relation is only visible for the top ten eigenvalues.

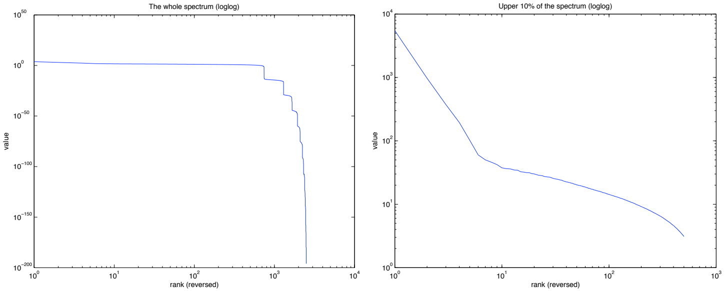

Figure: An example spectrum of a graph consisting of 5000 vertices. The eigenvalues were sorted descending and plotted on a log-log plot. The left figure shows the whole spectrum, the right one only 10% largest eigenvalues. In this case however, 70% of the nodes of lowest degrees were cut off. Nevertheless the left linear part of the plot still behaves like $y = -2x$. The rightmost part of the spectrum (which corresponds to small eigenvalues) exhibits interesting "stair regime", which requires further investigation.

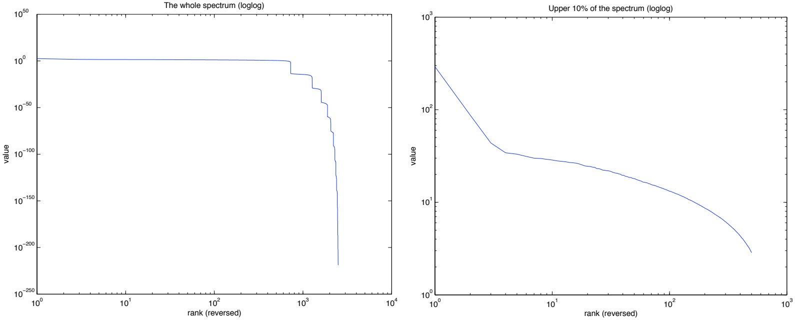

Figure: An example spectrum of a graph consisting of 5000 vertices. The eigenvalues were sorted descending and plotted on a log-log plot. The left figure shows the whole spectrum, the right one only 10% largest eigenvalues. In this case however, 70% of the nodes of lowest degrees and 1% (50) of the nodes of highest degrees were cut off. A couple of top eigenvalues behave like $y = -2x$. The "stair regime" is clearly visible at right of the spectrum.

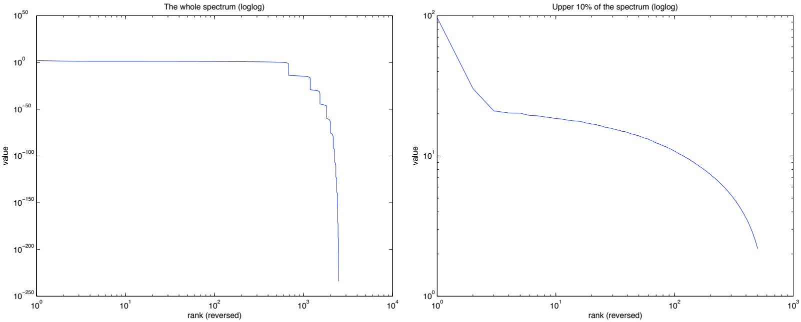

Figure: An example spectrum of a graph consisting of 5000 vertices. The eigenvalues were sorted descending and plotted on a log-log plot. The left figure shows the whole spectrum, the right one only 10% largest eigenvalues. In this case however, 70% of the nodes of lowest degrees and 3% (150) of the nodes of highest degrees were cut off. Here the $y = -2x$ becomes irrelevant, which suggests that the property is related to the elite part of the graph which has been fairly cut off.

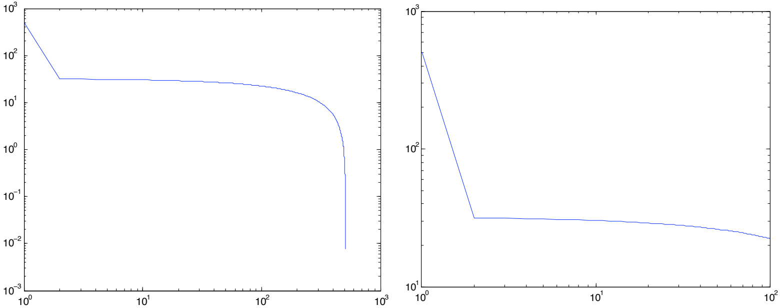

Figure: An example spectrum of a random graph consisting of 1000 vertices. There is one large separated eigenvalue and no sign of $\sim \frac{C}{k^2}$ relation.

Important remarks

- The theoretical results were obtained only for the truncated regime of the asymptotic Winner Take All (WTA) version of the spike flow model, and consequently it is not at all obvious whether and if so, to what extent the spectral footprint will be visible in finite, full blown model.
- The spectrum of the full blown model is messed by the large number of bulk units of low support which do not obey the WTA dynamics.
- Even though the $\sim \frac{C}{k^2}$ regime is observable only for a few top eignevalues, it has significant share in total spectrum "mass" (since all other eigenvalues that do not follow the relation are very small)

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Conclusions and future work

- The spectral footprint combined with degree distribution and other characteristics could be used to study similarity between networks.
- The spike flow model introduced previously has a clearly distinguishable spectral characteristics as imposed by both theoretical and numerical results.
- There are variants of the spike flow model equipped with geometrical features (presented here is a mean field model), which seem to share a lot of features of the original version (work in progress).
- The next step is to actually analyze the empirical functional networks recorded in various situations and methodologies (work in progress).

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Questions and Answers

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