Mesoscopic Approach to Locally Hopfield Neural Networks in Presence of Correlated Patterns

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Abstract— **Hopfield networks have gathered a lot of attention in computer science in recent years, as they have the ability to model many interesting phenomena that occur in brains and complex physical systems, and yet the model is nice in analysis. In this paper we investigate a simple Hopfield network organised in a two dimensional mesh, with localised interactions. The network remembers a number of periodically repeated, spatially correlated patterns. The weights are obtained via Hebbian learning rule combined with some extra information about the structure of correlations between the patterns, that is our system is in the so called phase coexistence regime in which the free energy for all of the patterns is equal (none of the patterns dominates in the sense it is the unique minimiser of the free energy). The number of remembered patterns is well below the memory limits to simplify the analysis and avoid any network's capacity problems, we can therefore say the network is in finite loading regime. We argue that such a system can be accurately analysed in mesoscopic scale, in which it displays some phenomena characteristic for systems with large scale, isotropic interactions (e.g. Kac potential systems near Lebowitz-Penrose limit) like sharp phase interfaces, motion by mean curvature etc.**

I. INTRODUCTION

Assume we have a 2d pattern recognising network (content-addressed associative memory) and patterns $P_1 \ldots P_n$. The weights in such a network are given by

$$w_{i,j} = \sum_{k=1}^{n} \alpha_k P_k[i] P_k[j] \qquad (1)$$

where $\alpha_k \in [0, 1]$, $P_k[i] \in \{-1, 1\}$ and

$$\sum_{k=1}^{n} \alpha_k = 1 \qquad (2)$$

The α parameters balance the strength between patterns. They are often chosen as simply $\alpha = \frac{1}{n}$ although as argued in [11] that is not particularly the best thing to do if patterns P_k are correlated spatially[1].

To avoid degeneracies, throughout this paper we make the natural assumption that contributions to $w_{i,j}$ coming from different patterns are linearly independent, in particular this excludes the degenerate case of two patterns being negatives of each other. At a non-zero temperature β^{-1} the network evolves by flipping the state of a randomly chosen unit σ_i with probability 1 if it did not agree with the sign

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[1]For readers convenience we will briefly recall some of the arguments in this paper as well.

of the local field $S_i := \sum_j w_{i,j} \sigma_j$ and with probability $\exp(-2\beta|S_i|)$ otherwise, with $\beta \geq 0$ referred to as the inverse temperature. This is the so-called Glauber dynamics. Recall that under this dynamics the system converges to the equilibrium distribution $\mathbf{P}(\bar{\sigma}) = \frac{1}{Z} \exp(-\beta E(\bar{\sigma}))$, where $E(\bar{\sigma}) := -\frac{1}{2} \sum_{i,j} w_{i,j} \sigma_i \sigma_j$ is the so-called energy function whereas Z, often referred to as the partition function, stands for the normalizing constant. In the sequel, we shall always assume that β is sufficiently large, i.e. that the system is in *low temperature regime*.

In practice one usually assumes that $\alpha_i = \frac{1}{n}$. This is correct in principle as long as patterns $P_1...P_n$ are independent, which is rather seldom satisfied in real world applications. In general, with the patterns more or less correlated the parameter vector α should be chosen in the way that would re-establish their joint stability – that is to say, as made precise below, $(\alpha_1, \ldots, \alpha_n)$ should coincide with the phase coexistence point. To put it in intuitive terms, in the presence of non-negligible correlations it is to be expected that, in the course of the retrieval process, the patterns which are more similar to each other will form 'coalitions' against those exhibiting weaker correlations, possibly making them unstable. This phenomenon can be compensated for and prevented by adjusting the parameters α and by assigning larger α to weaker patterns. It was shown in [11] that such choice of α is uniquely determined by the patterns and the temperature of the system. Assuming β to be large enough, we can use the Pirogov-Sinai theory to characterize the joint geometry of stability regions for all subsets of the pattern collection $\{P_1, \ldots, P_n\}$. Note that in the particular zero-temperature case the problem of choosing α has been solved and an iterative algorithm combining both the features of Hebbian rule and simple perceptron learning is available for training Hopfield-type networks in this setting, see [3] (learning rule I) and the references therein. As argued in [11] this does not have to be the correct choice for non-zero temperatures though.

It should be noted that, our research being focused on the influence of the correlation structure upon the pattern stability, we work in strictly finite loading regime, with a relatively small number of patterns memorized by large networks, thus avoiding any network capacity problems as falling beyond the scope of the present article.

To represent strong and regular spatial correlation structure we require our patterns to be periodic, that is to say the

pattern in a large region splits into a collection of *basic squares* containing periodic repetitions of some *basic pattern*. Moreover, we localize the connections, that is to say each unit is connected only to units lying at lattice sites within a certain given distance. These assumptions place our model within the general framework of lattice Gibbs measures, see [4]. For a finite number of memorized patterns this puts us in a position to apply the mathematical Pirogov-Sinai theory providing qualitative characterization of low temperature phase diagrams[2], see [14], [15]. To formulate the results of this theory, we introduce the following concept of pattern stability. For a given pattern P and a given large rectangular region R we fix the contents of the boundary basic squares to coincide with the basic pattern corresponding to P. Next, we run the Glauber dynamics for the interior neurons while keeping the states of the boundary units *frozen*, until the system is close to thermodynamic equilibrium. In low enough temperatures the structure of this equilibrium is often the following: the domain R contains an *ocean* of sites whose states agree with the pattern P surrounding isolated and small *disagreement islands*. The mathematical theory formalizes it in infinite volume limit stating that under appropriate conditions all the disagreement islands are finite and surrounded by the infinite and connected agreement ocean. If the equilibrium measure for boundary condition P exhibits such a structure, we call it the *pure phase* corresponding to P and we say that P is *stable*, otherwise we say that pure phase corresponding to P does not exist and we declare P *unstable*. Note that in this setting a pattern is stable if it can be retrieved only from the boundary information (frozen in the course of system evolution), which is stronger than the usually adopted definitions, see Section 4.1 in [10]. In this setting, for sufficiently large β, modulo certain technical details falling beyond the scope of this article, the following statements are direct conclusions of the Pirogov-Sinai theory, see [14], [15].

- In the space of parameters α, $\alpha_k \in [0,1]$, $\sum_{k=1}^{n} \alpha_k = 1$ there exists a unique choice of $(\alpha_1, \ldots, \alpha_n)$ under which all patterns $P_1, \ldots, P_n$ are stable. We say that all k pure phases co-exist at this point.
- For each subset $\pi \subseteq \{1, \ldots, n\}$ of cardinality $n-1$ there exists a smooth curve in the parameter space α marking the coexistence of phases $\{P_i \mid i \in \pi\}$. All these n curves meet at the point of coexistence of all phases.
- In general, for $k \geq 1$ and for $\pi \subseteq \{1, \ldots, n\}$ of cardinality k there exists an $n-k$-dimensional smooth hypersurface in the parameter space α marking the coexistence of phases for $\{P_i \mid i \in \pi\}$. Moreover, hypersurfaces corresponding to π and π' meet at the hypersurface marking the coexistence of phases with indices in $\pi \cup \pi'$.

[2]Note that Pirogov-Sinai theory covers systems in thermodynamical limit (infinite), while our system is evidently finite, so it is not *a priori* obvious to what extent experimental result would match with theoretical predictions. It seems however, as argued further in the paper, that the considered system size is large enough to clearly observe effects predicted by the theory.

This means in particular that for most choices of $(\alpha_1, \ldots, \alpha_n)$ **exactly one pattern is stable** and only under careful choice of α do we hit phase coexistence hypersurfaces. Clearly, the purpose of the considered networks being to memorize and retrieve all patterns, we should aim at identifying the phase coexistence point for all patterns $P_1, \ldots, P_n$ and setting the values of $(\alpha_1, \ldots, \alpha_n)$ accordingly.

We conclude this section by noting that, as already mentioned, all our considerations are specialized to the case of β large enough (low temperature regime). This is due to the fact that at a certain inverse temperature β, depending on the set of memorized patterns, an order-disorder phase transition occurs and at high enough temperatures the system starts ignoring the boundary conditions and producing disordered random mixtures of all stable patterns rather than trying to retrieve a particular one, see e.g. [5] or Lectures 5,6,7 in [9].

Fig. 1. Examples of correlated patterns used in the simulation. The leftmost pattern is the strongest one (it exhibits the lowest energy surplus). The second one from the left is highly correlated to the first one and it forms a 'coalition' against the third one , exhibiting the lowest correlations with other patterns. The fourth pattern has rather low correlations with the other ones as well.

A. Model details

In this paper we will focus mainly on the phenomena observed in simulations for a choice of α such that the system is approximately in the phase coexistence regime. As mentioned before, establishing exact location of phase coexistence point in general case is not trivial[3]. One can however easily derive appropriate α in zero temperature [8], by solving a system of linear equations, but as the temperature rises the coordinates of phase coexistence point change. For the scope of this article we will be mostly satisfied with the zero temperature solution since the temperature of examined system is low, however further in the paper we show that a slight deviation from the exact solution can lead to observable effects. As suggested before these issues have been described in details in [11], some examples of phase diagrams and numerical approximations of phase coexistence point have also been provided. Yet there is a simple technique that enables us to model phase coexistence regime in non-zero temperatures, that is the interactions of stable patterns and their negatives. Note that in the absence of external field interactions between a pattern and its negative are implicitly balanced, independently of the α parameters (stability of a pattern imposes stability of its negative).

We assume that each neuron in our system interacts with surrounding neighbours in range of four neurons (figure 2).

[3]There are no efficient algorithms available for the problem, and some expressions that can be derived from Pirogov-Sinai theory have exponentially growing complexity of coefficients.
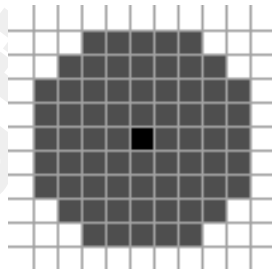
Fig. 2. Graphical representation for the range of interaction in considered model. Each pixel in the grid corresponds to a neuron. The black neuron in the middle interacts only with those gray.

We have chosen this radius of interaction as big enough to be able to analyze system in terms of mesoscopic theory, yet small enough to save computational power and be able to quickly reach approximate thermodynamical equilibrium. The consequence of such small range of interaction is the fact that the system is not perfectly isotropic and some disturbance by pattern structure may occur (the simulations have shown however, that the disturbance is not large, since the patterns are themselves quite small in comparison with the range of interactions). We can also model isotropic behaviour by introducing patterns that are approximately invariant to rotation (for example pattern $\mathcal{P}_1$ and $\mathcal{P}_2$).

Note that all our considerations are provided in finite loading regime, that is the number of remembered patterns is relatively small and we don't have to worry about the network's capacity. In fact increasing the number of patterns would lead to a significant decrease of the critical temperature where the order/disorder phase transition occurs.

The model we consider consists of 250x250 neurons, organized on a 2d lattice. The network is trained with rule described in 1. The global system is trained to remember periodic repetitions of small 5x5 patterns (basic cells). That way we can observe global phenomenon of pattern retrieval. The network has a fixed boundary condition given by repetitions of one of the basic patterns. This simulates thermodynamical limit (infinite extension of the system), so we simulate just a small portion of a hypothetical infinite system. The dynamics in our simulation is Asynchronous Glauber dynamics. The network evolves by flipping spins of randomly chosen units, with probability 1 if they did not agree with the sign of local field $S_i = \sum_j w_{i,j}\sigma_j$ and with probability $exp(-2\beta S_i)$ otherwise, with $\beta > 0$ referred to as inverse temperature.

II. MOTIVATIONS

The aim of the simulation was to check if the system can be described in terms of mesoscopic scale theory, therefore we are not interested in the state of a particular neuron, but a joint state of whole groups of neurons neglecting all the statistical disturbances. This allows us move from statistical mechanics of the model into continuos mechanics, and provide some expressions describing interesting features of the evolution in terms of differential equations. As shown below, in low temperature and mesoscopic scale the system

exhibits quite nice continuos behaviour that seems quite general. Unfortunately, precise mathematical proofs of such phenomena are rather complicated and were provided for specific classes of systems, like Kac potentials. Even though our model is not strictly a Kac potential model, we well briefly recall some of the key ideas of Kac potential theory, as we argue in the paper it can be a qualitatively good theoretical description of the behaviour observed in our simulation. Kac potentials have been introduced by Kac, Uhlenbeck and Hemmer [7] in the 60's for modelling, in the framework of statistical mechanics, the van der Waals theory of phase transitions. The main idea in this theory is scaling. There are three basic scales in the mentioned system: lattice distance, interactions range and size of the system. On one hand we have Ising models in which the range of interactions matches lattice distance (for Ising systems we have some nice theoretical results, proofs of phase transitions by Peierls argument [12] and other classical outcomes) , on the other hand we have mean field systems in which the range of interactions matches the size of the system (these systems provide phase transitions easily but are considered as non physical since the existence of phase transition does not even depend on the system's dimension) . Between these two extreme situations we have systems described as those with "large but finite range of interactions".

Definition 2.1: Kac potential (informal). Assume we have a function $J : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ that satisfies:

- $J(r, r') = J(r + a, r' + a) \geq 0$ for all r, r' and $a \in \mathbb{R}^d$
- $J(0, r)$ is continuous with compact support and normalised as probability kernel

$$\int_{\mathbb{R}^d} J(0, r)dr = 1 \qquad (3)$$

Consider a system with coupling constants satisfying:

$$J_\gamma(x, y) = \gamma^d J(\gamma x, \gamma y) \qquad (4)$$

for $\gamma > 0$. We say that such a system has a Kac potential.

In the above definition we omit a lot of important details like the definition of energy function, for precise definitions see chapter 3 in [13]. A lot of theoretical considerations about Kac potentials are done in the so called Lebowitz-Penrose limit with $\gamma \to 0$. This limit doesn't have any straightforward physical meaning, it is more a tool used on theoretical ground to provide proofs of certain theorems like the existence of phase transitions etc. We can however think of a system that is close to Lebowitz-Penrose limit, as of a system in which the range of interactions is sufficiently large to exhibit behaviour that has been proven to take place in the limit. As we argue in further sections, numerical experiments show that it is not very difficult to get "near" the Lebowitz-Penrose limit with the system we examine even though a range of four neurons in our model does not seem "large" at first glimpse. Another important detail is that formally Kac Potentials are defined for systems where there are two opposite states of magnetisation whereas we allow three or more patterns. This flaw can be overcome by introducing

generalised Kac potential in systems with more complex structure of spins (this requires some tricks in refinement of the Hamiltonian but is rather painless).

III. RESULTS

As mentioned before we analyse the evolution of the system in temperature low enough, to expect stable phases (energy wins against entropy). Furthermore, we try to hit the phase coexistence point by solving linear equations of energy as follows:

$$
\begin{cases}
E(P_1|\alpha_1, \alpha_2, ..., \alpha_n) & = E(P_2|\alpha_1, \alpha_2, ..., \alpha_n) \\
E(P_2|\alpha_1, \alpha_2, ..., \alpha_n) & = E(P_3|\alpha_1, \alpha_2, ..., \alpha_n) \\
& \vdots \\
E(P_{n-1}|\alpha_1, \alpha_2, ..., \alpha_n) & = E(P_n|\alpha_1, \alpha_2, ..., \alpha_n) \\
\alpha_1 + \alpha_2 + ... + \alpha_n & = 1,
\end{cases}
\tag{5}
$$

where $E(P_x|\alpha_1, ..., \alpha_n)$ is the 'specific energy' of pattern P_x for parameters $\alpha_1, ..., \alpha_n$. Since the energy is defined by

$$
\begin{aligned}
E(\bar{\sigma}) &= -\frac{1}{2} \sum_{i,j} \sigma_i \sigma_j w_{i,j} = \\
&= -\frac{1}{2} \sum_{i,j} \sigma_i \sigma_j \sum_{k=1}^{n} \alpha_k P_k[i] P_k[j],
\end{aligned}
$$

the above expression can be expanded as follows

$$
\begin{aligned}
E(P_x|\alpha_1, ..., \alpha_n) &= \\
&= \alpha_1 E(P_x|1, 0, ..., 0) + ... + \alpha_n E(P_x|0, ..., 1)
\end{aligned}
$$

and appropriate solution $[\alpha_1, ..., \alpha_n]$ can be provided. Unfortunately as argued in [11] phase coexistence point drifts as temperature changes, and this linear approximation can be considered rough (especially in medium and high temperatures). To overcome this restriction we use negatives to investigate behaviour in strict phase coexistence regime, since we do not have an external field. The experiments have shown that the system is big enough to clearly exhibit some interesting behaviour and moreover some other effects related to the α vector and its position against the phase coexistence point have been revealed.

A. Experimental procedure

All the simulations in this paper were performed as follows:

- A set of patterns was chosen. In some cases it included all four patterns (See fig. 1), and in some there was only one pattern (pattern/negative case).
- In case of many patterns, an approximate phase coexistence regime was established by solving the zero temperature approximation (equation 5)
- The network was set to its initial state:
  - Random, with boundary condition set as one of the remembered patterns.
  - Filled up (including the frozen boundary) with a chosen pattern with a blob of another (in most cases stable) pattern inside.
- The simulation was then carried out, with low temperature Glauber dynamics.
- In the experiments with shrinking phase interfaces, an estimate number of iterations necessary to have a blob of one pattern inside the other vanished was computed (data for the plots 6, 7).

B. Bulk phenomena

For a choice of α where one pattern strongly dominates, some bulk phenomena can be observed. The process of consuming a blob of weak pattern becomes rapid and occurs all over its volume. The existence of this effect is quite obvious and can be clearly observed in the course of the simulation (fig. 3).
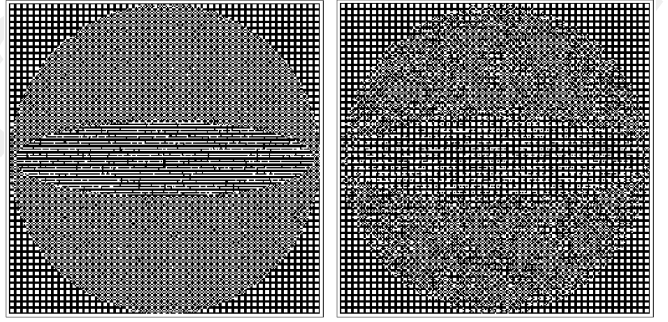

Fig. 3. Network's evolution in the regime of strict domination of one pattern, at a medium-low temperature ($\beta = 0.2$), at the initial stage of simulation (left) and after $9 \cdot 10^4$ steps. Ellipse and circle of unstable patterns are instantly consumed by the dominating stable one. This bulk phenomenon is very rapid in contrast to slow boundary evolution discussed further in the paper.
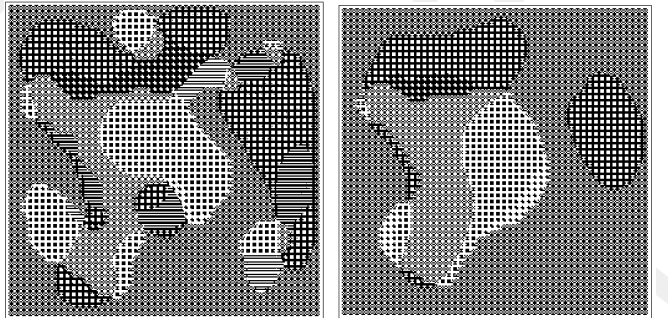
C. Boundary phenomena

Fig. 4. Network's evolution in the phase coexistence regime in different stages (random initialisation). One of the patterns is favoured in the sense its being set as the boundary condition (therefore, eventually it will dominate on the whole board). One can notice the spots of other stable patterns (as well as their negatives) sharing domain. Please note that the boundaries between the spots are sharp. Another interesting detail is that these spots seem to overlap.
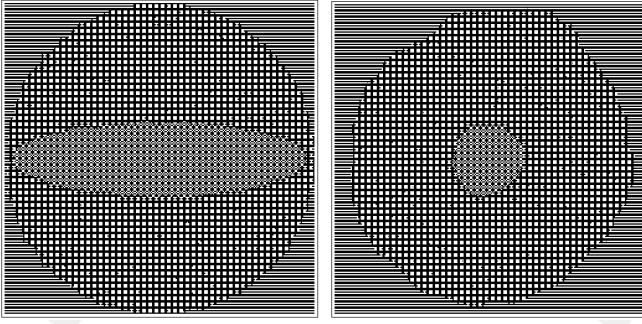
Fig. 5. Network's evolution in the coexistence regime. Please note the stable sharp interfaces between the patterns, as well as the fact that during the simulation the ellipse in the middle has turned into a small circle, which suggests movement by curvature. These phenomena are slow and stable in contrast with those from figure 3.

More interesting effect that we can observe is shrinking by mean curvature of the phase boundaries (interfaces between the regions dominated by patterns that occur during the simulation). Motion by mean curvature is common in nature, for example bubbles of gas scattered in liquid change their shapes according to mean curvature of their surfaces (it is one of the reasons why bubbles are spherical). Fluid floating in gas (or vacuum) in zero gravity is another example of such behaviour. Note that these examples have something else in-common with the issues we consider here - they are examples of phase interfaces (gas/fluid).

Definition 3.1: Movement by mean curvature. A family of bounded, open smooth sets Λ_t in $\mathbb{R}^d$ moves smoothly by mean curvature with velocity $\nu > 0$ in the time interval $[0, T]$, $T > 0$, if for any $t \in [0, T]$, the boundary Γ_t of Λ_t is smooth and its points have normal (directed towards the exterior of Λ_t) velocity

$$v = -\nu\kappa$$

where κ is mean curvature calculated as positive if the concavity of Γ_t is towards its interior.

We consider 2d system so there is only one curvature parameter, therefore we will be calling this phenomenon "shrinking by curvature" instead of "shrinking by mean curvature" which is more general and assumes more dimensions and curvature parameters. The model we consider is relatively small (250x250 neurons, 50x50 basic pattern-cells), whereas the mentioned effects are considered in big systems where the range of interaction is insignificant in comparison to the size of the system and the size of a single particle is insignificant in comparison to the range of interaction (therefore such systems are "almost" continuos). We decided however to run the simulation hoping to observe some qualitative effects. First of all we have to note that we can observe sharp phase interfaces, which is absolutely necessary to expect any shrinking phenomena (See fig. 4). The shrinking phenomenon (as well as sharp interfaces) occurs only in the phase coexistence regime and, as mentioned before, determining the phase coexistence point in general

is a difficult problem unless we consider zero temperature system. In order to overcome these restrictions we decided to perform two simulations:

- In medium-low temperature with only one pattern and its negative - as mentioned before phases of a pattern and its negative are always mutually stable.
- In very low temperature with four patterns, assuming that the phase coexistence point is well approximated with the result achieved in the zero temperature. From other considerations [11] we know that the movement of phase coexistence point in the low temperatures regime is not radical, however results presented in this paper show that even small deviation of the α vector from the precise position of phase coexistence point may result in noticeable perturbation in the behaviour of the system.

Note that the shrinking by curvature implies that the radius of circular spot of one pattern inside the other satisfies

$$dr = -\lambda \cdot c(r)dt = \frac{-\lambda}{r}dt \qquad (6)$$

Where r is the radius of the spot and $c(r) = \frac{1}{r}$ is the curvature. From the above expression we derive

$$\frac{r^2}{2} = c - \lambda \cdot t \qquad (7)$$

That implies that the relation between the initial radius (R) of the spot and the time (number of steps T) needed for this spot to vanish should be strictly quadratic ($T = \epsilon R^2$) and this we can easily check experimentally. The first simulation revealed perfectly quadratic relation as predicted (see fig 6). The empirical data can be very accurately approximated with ax^2 parabola, by least squares method. Simple differential analysis confirms the quadratic character of the relation. In the second simulation we hit on something interesting (see fig. 7). The relation again seemed quadratic, but with non negligible linear part (the data was well approximated with $ax^2 + bx + c$ function with non-negligible b, whereas the approximation with $ax^2 + c$ was quite poor). We claim to have an explanation for this phenomenon. If we assume that the relation is in fact quadratic with relevant linear part, then consequently we argue that the equation 6 should be rewritten as follows:

$$dr = -\lambda \cdot c(r)dt + \mu = \frac{-\lambda}{r}dt + \mu \qquad (8)$$

for some non zero parameter μ. That suggests that the speed of shrinking depends on curvature and some other unknown parameter. A more thoughtful investigation of the observed inconsistency lead us to conclusion that the fact of not being exactly in the phase coexistence point might have an impact on the result of the experiment. Note that the second simulation was carried out in the approximate coexistence of four pattern phases. The phase coexistence point was extracted from linear equations of energy that are valid in the zero temperature regime. From the considerations in [11], we know that the phase coexistence point tends to drift as temperature rises in such a manner that strong
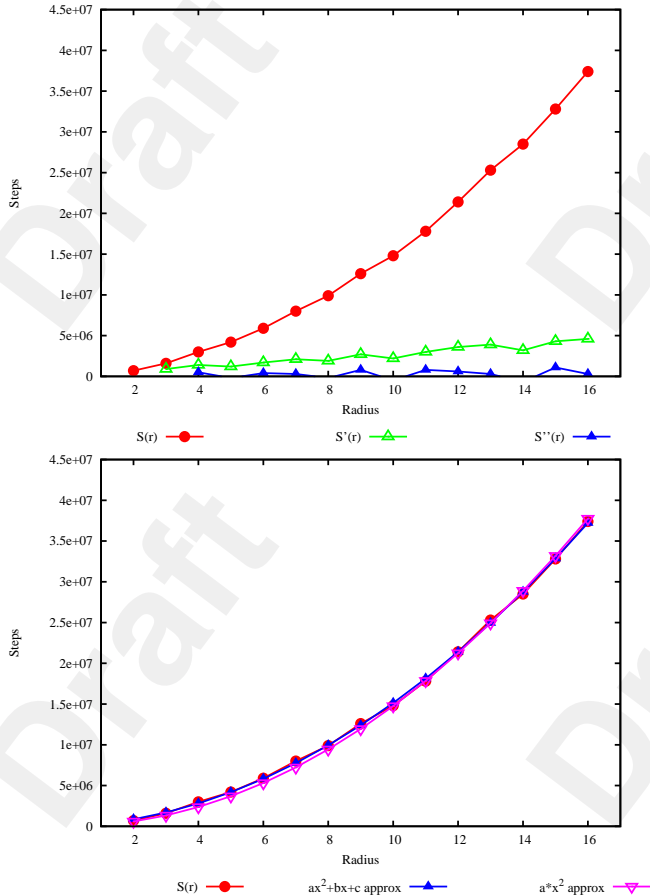
Fig. 6. Relation between number of iterations needed for circular spot of the pattern's negative to vanish in the ocean of a pattern and the radius of mentioned spot. The top figure shows also first and second derivatives. Note that the second derivative (blue) is approximately constant what suggests that the relationship is quadratic. The bottom figure shows two parabolic curves ($a \cdot x^2$ and $a \cdot x^2 + b \cdot x + c$) obtained from the least squares method, which perfectly approximate empirical data.
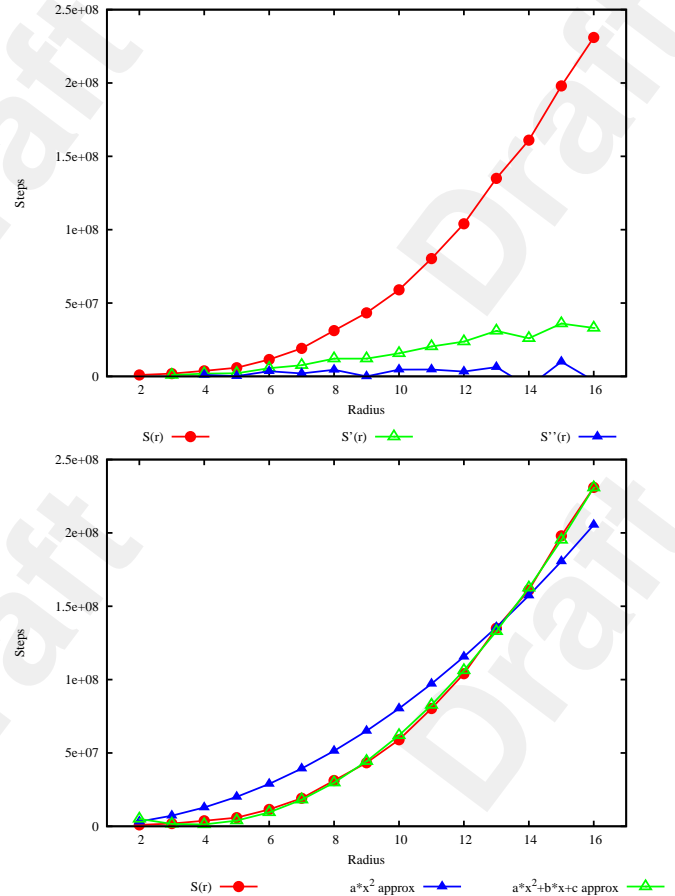
Fig. 7. Plots similar to those from figure 6 obtained from the second simulation. Note that the relation is still rather quadratic, but the ax^2 parabola (blue) gives significantly worse results in approximating empirical curve than the $ax^2 + bx + c$ parabola, what implies relevance of the linear part of the relation.

patterns are getting even stronger, while the weak patterns are getting weaker[4]. In our simulation the surrounding pattern was the strong one, while the one inside the spot was weaker. Even though the bulk effects typical for strong domination of one pattern were not observed, the dominating pattern gained some extra force in consuming the spot of the weaker one. The μ parameter in the equation refers to that extra force. Note that actually if our model was big enough (say 10000×10000 neurons) we would observe bulk phenomena for the same parameters (but in a respectively larger time scale).

IV. CONCLUSIONS

The experiments we described in the paper provide numerical evidence for conjecture, that Hopfield networks with localised interactions under appropriate conditions (low temperature and phase coexistence regime) can be analysed in terms of mesoscopic theory well approximated by Kac potentials. Even though our system is quite small and the range

[4]"stronger/weaker" as described in the introduction.

of interactions does not seem large at all, it clearly exhibits phenomena like sharp phase interfaces and their dynamical behaviour like shrinking by mean curvature. Some other interesting phenomena occurred when the simulation ran in a vicinity of the phase coexistence point, where movement by curvature gains some extra force originating from the broken symmetry between the patterns. On the other hand, when one of the phases strictly dominates, the behaviour proves to be completely different and reveals instant bulk nature. In the scope of these results is seems reasonable to make an assumption that some activations in brains could evolve (with respect to topology of neural connections) in a way described by mechanics of phase interfaces, which by itself seems quite interesting, but beyond question requires additional examination.

The mesoscopic approach has certain advantages over the reductionist analysis, since neurodynamics of compound neural systems can be enormously complex. The analysis in the "medium" scale lets us neglect all the (possibly unimportant) details and extract the significant facts about the investigated models, and provide an insight into interesting phenomena that occur in bigger scales. Unfortunately, there

is no general mesoscopic theory for neural networks, and it seems that appropriate descriptions can only be done in very special conditions like phase coexistence regime, in systems that are close to stationary distribution. Despite these difficulties, proposed analysis can lead to completely new models that would possibly obtain the same results a lot faster in bigger scales.

There are however some unanswered questions that leave a field for further research, that includes developing an efficient algorithm for establishing the precise coordinates of the phase coexistence point in non zero temperatures, as well as generalising these results for networks with more complex structure of connections, possibly small world networks etc.

V. ACKNOWLEDGEMENTS

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