Robustness of Power Laws in Degree Distributions for Spiking Neural Networks

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***Abstract*—Power law graphs are an actively studied branch of random graph theory, motivated by a number of recent empirical discoveries which revealed power law degree distributions in a variety of networks. Power laws often coexist with some degree of self-organization either based on growth and preferential attachment (which seems to be the case in sociological/technological networks) or duplication (which seems to be the case for biological/methabolic networks). Quite recently a power law graph with exponent $\gamma \approx 2$ has been observed in fMRI brain studies of correlations of functional centers of activity. We study the model we introduced previously to explore possible mechanisms existing in large neural networks that might lead to power law connectivity. The model (referred to as *the spike flow model*) resembles a kind of spiking neural network and yields a power law graph with exactly $\gamma = 2$ as a byproduct of its dynamical behavior. In this paper we investigate whether the power law is robust under certain changes to the model's dynamics. In particular we study the effect of merging the model with a random Erdős-Rényi graph which can be interpreted as an addition of long range myelinated connections. Our numerical results indicate that as long as the density of Erdős-Rényi fraction is bounded by a constant, the power law is preserved in systems of appropriate size.**

I. Introduction

POWER LAW degree distributions are present in a variety of information processing networks and systems as discussed in [1] and [2]. Such power laws often coexist (but not imply as shown [3]) with certain structural features like the existence hubs, self-similarity under various operations (contractions, degree preserving rewirings etc.) and high level of resistance against random attacks. Since power law graphs emerge spontaneously in diverse networks ranging from the World Wide Web [4], science collaboration networks [5], citation networks [6], ecological networks [7], linguistic networks [8], cellular metabolic networks [9], [10] to telephone call network [11], [12], it is quite natural to ask whether neural systems could benefit from such an architecture, and if so, whether there are any mechanisms inherent to neural activity that might lead to a power law connectivity. The answer to the first part of the above question seems to be yes. Power law graphs are fairly well connected in comparison to corresponding (in terms of number of edges) random graphs and offer better communication between remote units (see [13], [14], [15] for studies of neural networks built on power law graphs). Quite recent empirical study conducted using fMRI [16], [17] shows that power law structures are relevant for modeling brain activity at medium level (excitations of groups of neurons giving rise to a functional network). These results are by no means in contradiction to the outcome of studies of C.elegans worm nervous system that showed exponential decay of degree distribution [18], [19]. The network of C. elegans is very small (the whole organism has only about 1000 cells) and the mechanisms we discuss seem to play important role on the higher level of (possibly large) groups of neurons, not on the level of single cells and synapses. These results give further motivation to study the power law connectivity in the context of neural networks, see [20] and references therein for a comprehensive survey on scale-free structures in neocortex on various levels and contexts. In our previous papers [21], [22], [23] we addressed the second part of the above question, concerning neural-like mechanism that might lead to a power law connectivity. Note that the original model of Barabási and Albert [24] based on growth and preferential attachment does not describe the situation considered in [16] since growth in this case is very limited. The other reason is that Barabási-Albert model in its most natural setup leads to power law exponent $\gamma = 3$ while empirical studies of [16] strongly suggest $\gamma = 2$.

The model introduced in [21], [22], [23] (referred to as the *spike flow model*) results in a power law network with $\gamma = 2$ simply as a byproduct of its dynamical behavior. Furthermore the model inherits many features of typical neural networks positioning itself somewhere between the classical Hopfield model [25] (or rather its stochastic variant [26]) and more complex spiking neural networks. The fairly complex state space and state memory make the *spike flow model* more adequate in modeling large groups of neurons with high feedback connections rather that single spiking neurons. In subsequent sections of this article we briefly recall the details of the investigated model and in further sections study the robustness of the power law against certain changes to the model's dynamics.

II. The spike flow model

The model consists of nodes σ_i, $i = 1 \dots N$. Each node's state is described by a natural number from some fixed interval $[0, M_i]$. In the scope of this paper we assume $M_i = \infty$, that is the state space is unbounded (when $M_i = 1$ on the other hand the model much resembles Hopfield network). The network is

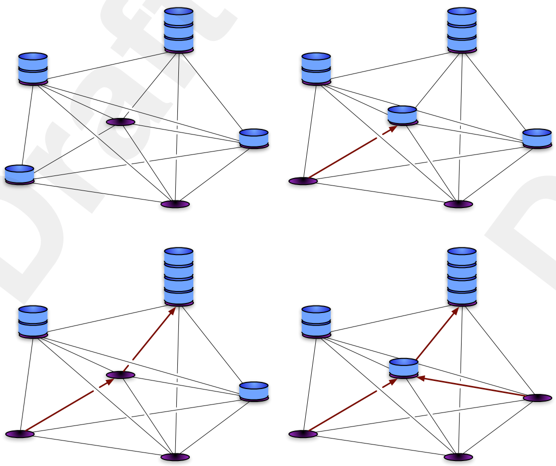

Fig. 1. Schematic description of the *spike flow model* (top left to bottom right) - the nodes contain some amount of tokens (units of charge), which can be exchanged under stochastic dynamics. An event of charge exchange can be seen as a spike (action potential) whence the name - *spike flow model*.

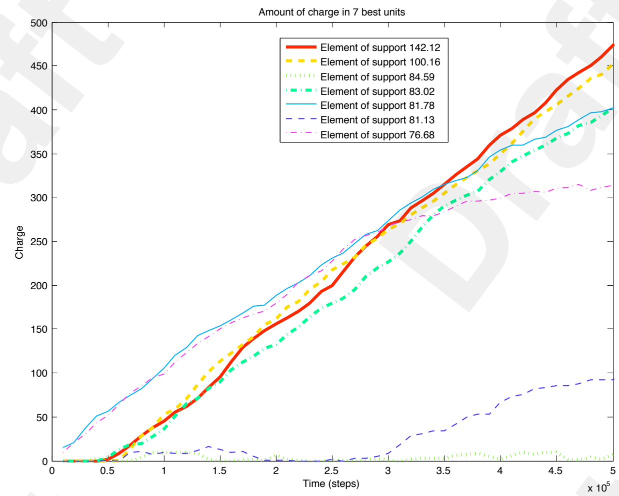

Fig. 2. Amount of charge in 7 best units in terms of support plotted against time (simulation steps) in the early stage of evolution of the original *spike flow model*. In this stage the *elite* drain the charge from the bulk at fairly equal rate.

built on a complete graph in that there is a connection between each pair of neurons $\sigma_i, \sigma_j,\ i \neq j$, carrying a real-valued weight $w_{ij} \in \mathbb{R}$ satisfying the usual symmetry condition $w_{ij} = w_{ji}$, moreover $w_{ii} := 0$. The values of w_{ij} are drawn independently from the standard Gaussian distribution $\mathcal{N}(0,1)$ and are assumed to remain fixed in the course of the network dynamics. The model is equipped with the Hamiltonian of the form:

$$\mathcal{H}(\bar{\sigma}) := \frac{1}{2}\sum_{i \neq j} w_{ij}|\sigma_i - \sigma_j| \tag{1}$$

if $0 \leq \sigma_i \leq M_i,\ i = 1, \ldots, N$, and $\mathcal{H}(\bar{\sigma}) = +\infty$ in the other case. Here $\bar{\sigma}$ denotes of the state of the whole system. The dynamics of the network is defined as follows: at each step we randomly choose a pair of neurons (units) $(\sigma_i, \sigma_j),\ i \neq j$, and denote by $\bar{\sigma}^*$ the network configuration resulting from the original configuration $\bar{\sigma}$ by decreasing σ_i by one and increasing σ_j by one, that is to say by *letting a unit charge transfer from σ_i to σ_j*, whenever $\sigma_i > 0$ and $\sigma_j < M_j$. Next, if $\mathcal{H}(\bar{\sigma}^*) \leq \mathcal{H}(\bar{\sigma})$ we accept $\bar{\sigma}^*$ as the new configuration of the network whereas if $\mathcal{H}(\bar{\sigma}*) > \mathcal{H}(\bar{\sigma})$ we accept the new configuration $\bar{\sigma}^*$ with probability $\exp(-\beta[\mathcal{H}(\bar{\sigma}^*) - \mathcal{H}(\bar{\sigma})]),\ \beta > 0$, and reject it keeping the original configuration $\bar{\sigma}$ otherwise, with $\beta > 0$ standing for an extra parameter of the dynamics, in the sequel referred to as the inverse temperature conforming to the usual language of statistical mechanics. In the present paper we will assume β fixed and large, that is the system is in low temperature regime and so such "stochastic" jumps are rare.

Note that in this setup positive weights $w_{i,j}$ favor agreement of states σ_i and σ_j, while negative weight favor disagreement.

Whenever a unit of charge is exchanged between two nodes that fact is recorded by increasing the counter associated with a corresponding edge (Fig 1). The edges (and nodes) being frequently visited by units of charge are in the focus of our interest. We refer to the resulting weighted graph as to the *spike flow graph*.

In [22] a number of results related to the *spike flow model* have been established:

- In contrast to a seemingly complex dynamics, with high probability there is a unique ground state of the system, in which all the charge is gathered in a unit that maximizes

$$S_i := -\sum_{j \neq i} w_{ij}. \tag{2}$$

  referred to as *support* in the sequel. The proof goes by a mixture of rigorous and semi rigorous calculations and has a rather asymptotic character, but is in full agreement with numerical simulations for systems containing between a couple hundreds to a couple of thousands of nodes.
- The system's behavior eventually admits a particularly simple approximation in terms of a kind of winner-take-all dynamics: almost all transfers converge to units of maximal support (referred to as the *elite* see Figure 2, while the others referred to as the *bulk*), which then compete in draining charge from each other. Ultimately the unit of maximal support gathers all of the charge and the system freezes in a ground state (Fig 3).
- The node degree distribution (where by degree we mean the sum of counters of edges adjacent to a given node[1]) obeys a power law with exponent $\gamma = 2$. The proof is based on the elite/bulk approximation and properties of ordering sequences. Again there is a strong agreement with numerical results (see Fig. 5)

[1]Since charge transfers are directed, we distinguish in and out degrees, but asymptotically these two are equal in terms of distributions.

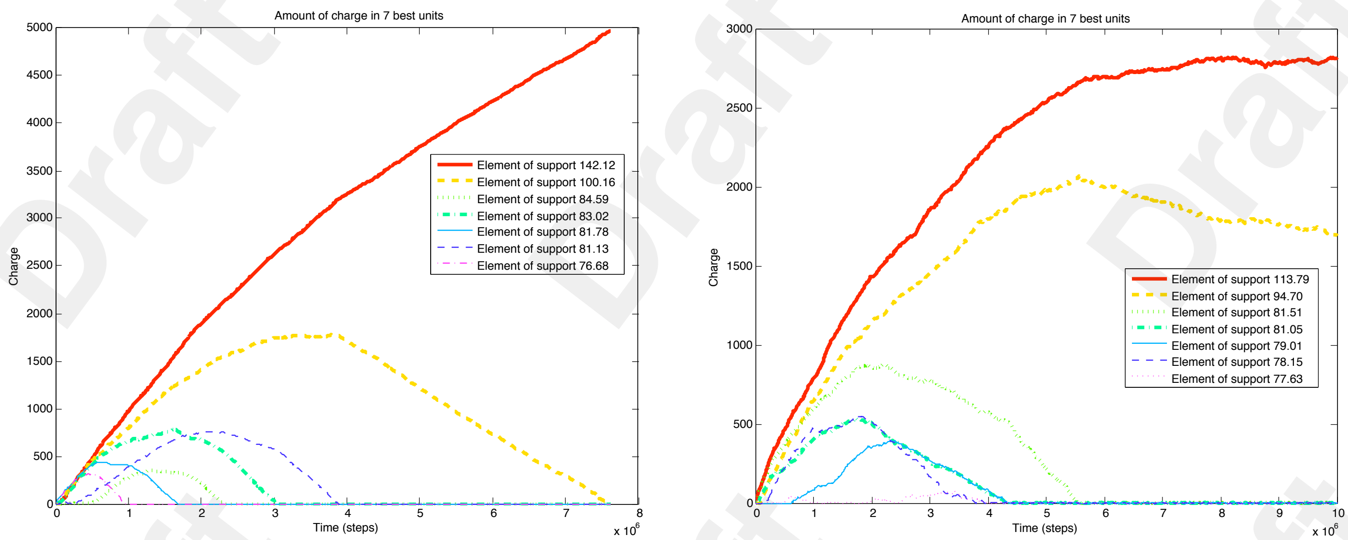

Fig. 3. Amount of charge in 7 best units in terms of support plotted against time (simulation steps). Left figure shows the evolution of an undisturbed spike flow model, the right one includes Erdős-Rényi random fraction with average of 10 edges/node. Clearly in both cases the elite emerges and drains most of the charge from bulk units, eventually pumping everything to the winner. In the right figure however one can see some discrepancies being the result of presence of Erdős-Rényi fraction.

In later work also the spectra of resulting spike flow graphs have been studied giving interesting results [27].

Before moving on to other parts of the paper it is necessary to give an interpretation of the presented model in terms of neural networks. Obviously biological neurons do not act like the units of the spike flow model, they cannot keep their excitation for later use and whenever they discharge (spike) they loose their memory about previous excitation (when the neuron reaches equilibrium state it essentially has zero "knowledge" of its previous activity). Things are different however, when a recurrent group of neurons is taken into account as a single computing unit. In this case it is possible to keep excitation within the group by loopback connections as suggested in [28] where (a bit artificial) model based on small recurrent groups composed of simple dynamical neurons [29] was investigated. The methodology of functional correlation graph construction in [28] quite accidentally resembles that in [16] and both result in $\gamma \approx 2$ power law. We also studied the properties of spontaneously emerging neuronal groups by repeating the results of [30] but in this case the outcome was not apparent (chapter 6 in [23]). The model in [30] contains long range delayed connections (reentrant connections) which introduce a lot of noise into the correlations of our interest. The present paper is aimed at introducing such long range connections to the *spike flow model* and investigating whether the power law of the spike flow model remains stable.

III. Numerical setup

The *spike flow model* is straightforward to implement. In every step the system's energy can be recomputed in linear time (there's no need to compute all the elements of the Hamiltonian (equation 1) since most of them don't change when only one unit of charge is exchanged. After a while most of the units (*the bulk*) are unoccupied and therefore another slight optimization becomes evident: instead of choosing the source node randomly among all of the units one can only choose from those non zero[2]. Subsequent unoccupied source node update attempts would be discarded anyway until an occupied unit has been hit. The choice of the destination unit is conducted among all units since there are no such restrictions in this case (we assume $M_i = \infty$, but if we had $M_i = const$ we could have used a similar trick). The numerical study of *spike flow model* has already been presented in [21] and [22]. In this article we focus on slight changes to the model's dynamics which are as follows:

- We merge the system with a random Erdős-Rényi graph [31], [32] whose set of vertices coincides with the set of units.
- If an edge between two units σ_i and σ_j happens to exist in the Erdős-Rényi fraction, and σ_i gets chosen as a source of charge transfer to σ_j, then the transfer gets accepted unconditionally, no matter what happens to the system's energy.
- The change into dynamics may seem a bit like increasing the temperature ($\beta = \frac{1}{kT} = 10$ was fairly large in our simulations, therefore generally the number of charge transfers related to stochastic mechanism acting against the energy factor is insignificant), but in this case only strictly fixed set of edges is allowed to act against the energy factor.

The random connections which disturb local self-organization may be interpreted as a set of long myelinated connections spanning from tightly connected local group of neurons to

[2] Its easy to verify that such a modification does not affect the spike flow distribution.

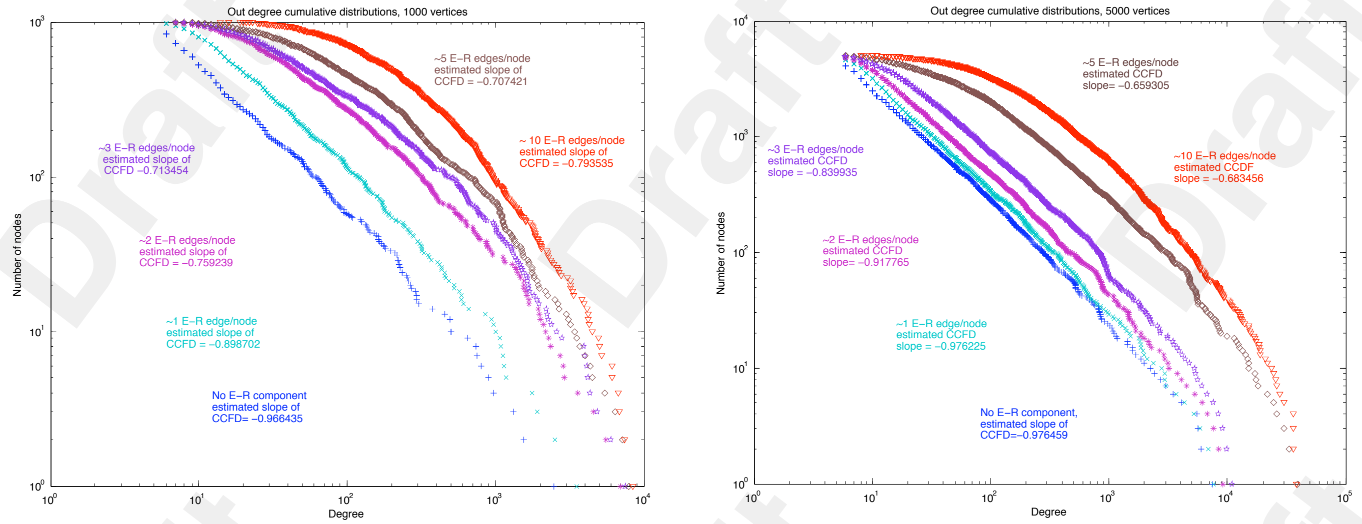

Fig. 4. Sample (out) degree distributions (complementary CDF, log-log plots) of systems with various density of Erdős-Rényi fractions. The left figure shows results from simulation of 1000 vertices, the right plot presents corresponding results from simulation of 5000 vertices. Clearly for the smaller system the presence of Erdős-Rényi fraction with approximately 2 edges/node already seriously disturbs the power law. The bigger system remains stable until the density of Erdős-Rényi fraction reaches approximately 5 edges/node.

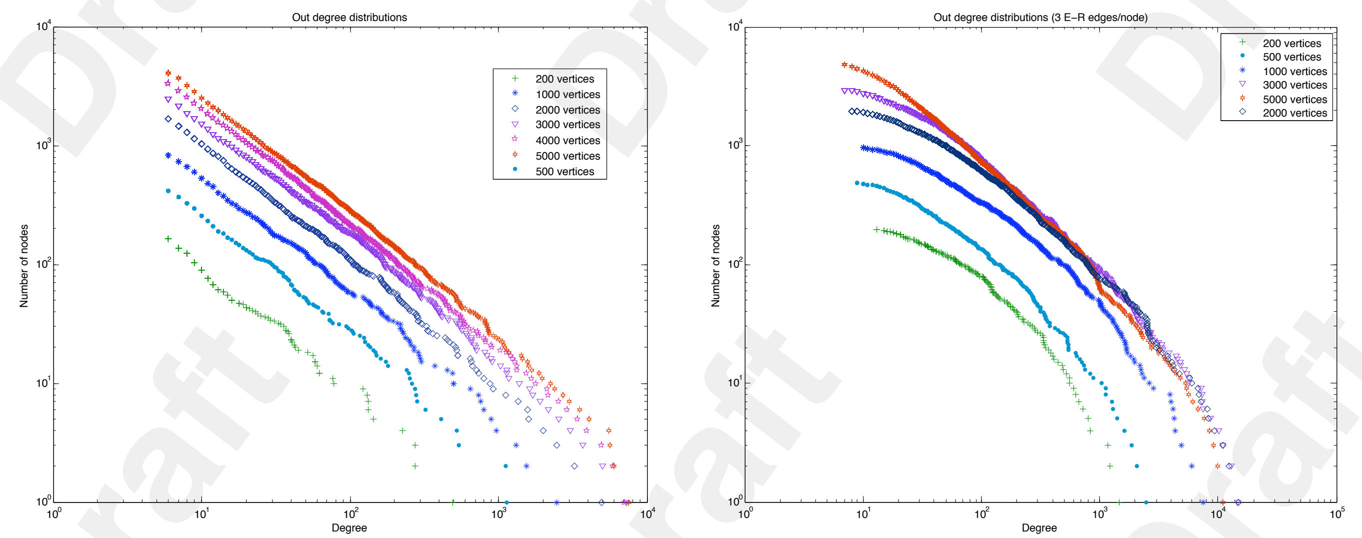

Fig. 5. Degree distributions (complementary CDF, log-log plots) of systems of various sizes plotted together. The left figure shows the original spike flow model, where the power law is well pronounced for a variety of sizes. The right one shows the model equipped with Erdős-Rényi fraction of approximately 3 edges/node. Note the curvature changes as the number of vertices gets increased. While small systems clearly reveal exponential decay, the distribution of a system of 5000 vertices (red) begins to resemble a power law.

some other such group placed in some more or less random parts of the ”brain”. The main question investigated in this paper is whether (and to what extent) the introduction of Erdős-Rényi fraction disturbs the winner-take-all dynamics and power laws. Simulations were carried for a number of instances of the spike flow model ranging from 200 up to 5000 vertices.

IV. Results

The simulations revealed a couple of interesting features. In all cases the introduction of Erdős-Rényi fraction whose density was less than one edge/node (below the giant component phase transition, see [33]) was negligible in terms of winner-take-all dynamics and power laws. More saturated Erdős-Rényi fractions however introduced disturbances into the dynamics and the degree distributions.

Figure 3 shows as the amount of charge in seven best units (in terms of support – equation (2)) changes with time. The left plot is based on the original *spike flow model* (an instance of 1000 vertices). The winner-take-all dynamics is evident (most of the interesting things happen in the first $3 * 10^6$ steps, when there is still fair amount of charge in the bulk units – see Figure 2 – afterward only a few units compete in charge exchange), the best unit eventually collects all of the available charge. The right figure shows corresponding plot of the *spike flow model* equipped with an Erdős-Rényi fraction with about 10 edges/node. Such a graph is quite dense in the discussed situation (it is with high probability connected and even if there are disconnected components, they are rather very small) and clearly it influences the dynamics, in particular the winner-take-all approximation. Even though the best unit eventually gathers most of the charge, the process is slower and non monotonic. Nevertheless the general idea of the *bulk* and the *elite* remains valid. With 1000 vertices in the spike flow model and 10 Erdős-Rényi edges/node the power law breaks down though (Figure 4, left). In fact for the system of 1000 vertices even adding Erdős-Rényi fraction with 2 or 3 edges/node significantly disturbs the power law distribution of node degrees. Things change when the system size is increased. Figure 5 shows how the node degree distributions change as the system size gets increased in the original *spike flow model* (left) and the model Erdős-Rényi fraction attached (we only plot complementary cumulative distribution functions – CCDF – since as argued in [3] frequency histograms can be misleading). In the latter case, the curvature of distribution (in terms of a log-log plot) decreases with the system size, converging to a power law (which is a straight line on a log-log plot). The simulation of 5000 vertices revealed that even adding Erdős-Rényi fraction with 3 edges/node leaves the power law nearly intact. Erdős-Rényi fraction with higher densities (5-10 edges/node) quite rapidly breaks down the power law in this case as well (again, see figure 4).

V. Conclusions

Summarizing the result we note that:

- Sparse Erdős-Rényi fractions added to the spike flow model leave the power-laws intact
- For more dense Erdős-Rényi fractions and small instances the power law breaks down
- The relative edge density at which the power law is broken depends on the size of the investigated system, i.e. the density which already breaks the power law in the system of 1000 vertices is still acceptable in the system of 5000 vertices.
- We conjecture that this is a general principle, and even bigger systems could allow for more dense Erdős-Rényi fractions, still retaining fair power law approximation

The existence of power law structures in various networks requires further study. In particular the existence of such structures in brain activity patterns might be valuable from medical/compuatational point of view. It is yet unclear whether power law structure in the brain is of fundamental importance or is it a byproduct of some other more fundamental mechanisms present in huge information processing networks. The *spike flow model* is aimed at shedding some light towards these issues. In this paper we presented numerical hints suggesting that the *spike flow model* is to some extent stable (in terms of winner-take-all approximation and power law degree distribution) under addition of *random* edges which are allowed to *act against the energy factor*. Since the *spike flow model* is mathematically tractable the next obvious step is to put together presented result into equations and provide a formal proof (work currently in progress). Other directions include supplying the model with topological features.

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References

[1] R. Albert and A.-L. Barabási, "Statistical mechanics of complex networks," *Reviews of modern physics*, no. 74, pp. 47–97, January 2002. [Online]. Available: http://arxiv.org/abs/cond-mat/0106096

[2] M. Newman, A.-L. Barabasi, and D. J. Watts, *The Structure and Dynamics of Networks: (Princeton Studies in Complexity)*. Princeton, NJ, USA: Princeton University Press, 2006. [Online]. Available: http://press.princeton.edu/titles/8114.html

[3] L. Li, D. Alderson, R. Tanaka, J. C. Doyle, and W. Willinger, "Towards a theory of scale-free graphs: Definition, properties, and implications (extended version)," Oct 2005. [Online]. Available: http://arxiv.org/abs/cond-mat/0501169

[4] R. Albert, H. Jeong, and A.-L. Barabási, "Diameter of the world-wide web," *Science*, vol. 401, pp. 130–131, Septmeber 1999.

[5] A.-L. Barabási, H. Jeong, Z. Néda, E. Ravasz, A. Schubert, and T. Vicsek, "Evolution of the social network of scientific collaborations," *Physica A*, vol. 311, no. 4, pp. 590–614, 2002.

[6] S. Redner, "How popular is your paper? an empirical study of the citation distribution," *European Physical Journal B*, vol. 4, no. 2, pp. 131–134, 1998.

[7] J. M. Montoya and R. V. S. V., "Small world patterns in food webs," *Journal of Theoretical Biology*, vol. 214, no. 3, pp. 405–412, February 7 2002.

[8] R. F. i Cancho and R. V. Solé, "The small-world of human language," *Proceedings of the Royal Society of London B*, vol. 268, no. 1482, pp. 2261–2265, November 7 2001.

[9] U. S. Bhalla and R. Iyengar, "Emergent properties of networks of biological signaling pathways," *Science*, vol. 283, no. 5400, pp. 381–387, January 1999. [Online]. Available: http://dx.doi.org/10.1126/science.283.5400.381

[10] H. Jeong, B. Tombor, R. Albert, Z. N. Oltvai, and A.-L. Barabási, "The large-scale organization of metabolic networks," *Nature*, vol. 407, no. 6804, pp. 651–653, October 5 2000.

[11] J. Abello, A. Buchsbaum, and J. Westbrook, "A functional approach to external graph algorithms," *ESA '98: Proceedings of the 6th Annual European Symposium on Algorithms*, pp. 332–343, 1998. [Online]. Available: http://dx.doi.org/10.1007/3-540-68530-8_28

[12] W. Aiello, F. Chung, and L. Lu, "A random graph model for massive graphs," in *STOC '00: Proceedings of the thirty-second annual ACM symposium on Theory of computing*. New York, NY, USA: ACM, 2000, pp. 171–180.

[13] J. I. Perotti, F. A. Tamarit, and S. A. Cannas, "A scale-free neural network for modelling neurogenesis," *Physica A Statistical Mechanics and its Applications*, vol. 371, pp. 71–75, Nov. 2006.

[14] D. Stauffer, A. Aharony, L. da Fontoura Costa, and J. Adler, "Efficient hopfield pattern recognition on a scale-free neural network," *The european physical journal B*, no. 32, pp. 395–399, 2003.

[15] P. N. McGraw and M. Menzinger, "Topology and computational performance of attractor neural networks," *Physical Review E*, vol. 68, p. 047102, 2003. [Online]. Available: http://www.citebase.org/abstract?id=oai:arXiv.org:cond-mat/0304021

[16] V. M. Eguíluz, D. R. Chialvo, G. A. Cecchi, M. Baliki, and A. V. Apkarian, "Scale-free brain functional networks." *Phys Rev Lett*, vol. 94, no. 1, January 2005. [Online]. Available: http://view.ncbi.nlm.nih.gov/pubmed/15698136

[17] O. Sporns, D. R. Chialvo, M. Kaiser, and C. C. Hilgetag, "Organization, development and function of complex brain networks." *Trends Cogn Sci*, vol. 8, no. 9, pp. 418–425, September 2004. [Online]. Available: http://dx.doi.org/10.1016/j.tics.2004.07.008

[18] L. A. Amaral, A. Scala, M. Barthelemy, and H. E. Stanley, "Classes of small-world networks." *Proc Natl Acad Sci U S A*, vol. 97, no. 21, pp. 11 149–11 152, October 2000. [Online]. Available: http://dx.doi.org/10.1073/pnas.200327197

[19] C. Koch and G. Laurent, "Complexity and the Nervous System," *Science*, vol. 284, no. 5411, pp. 96–98, 1999. [Online]. Available: http://www.sciencemag.org/cgi/content/abstract/284/5411/96

[20] W. J. Freeman, "Scale-free neocortical dynamics," *Scholarpedia*, vol. 2, no. 2, p. 1357, 2007. [Online]. Available: http://www.scholarpedia.org/article/Scale-free_neocortical_dynamics

[21] F. Piękniewski and T. Schreiber, "Emergence of scale-free spike flow graphs in recurrent neural networks," in *Proc. IEEE Symposium Series in Computational Intelligence - Foundations of Computational Intelligence*, Honolulu, Hawai USA, April 2007, pp. 357–362. [Online]. Available: http://doi.ieeecomputersociety.org/10.1109/FOCI.2007.371496

[22] ——, "Spontaneous scale-free structure of spike flow graphs in recurrent neural networks," *Neural Networks*, vol. 21, no. 10, pp. 1530 – 1536, 2008. [Online]. Available: http://dx.doi.org/10.1016/j.neunet.2008.06.011

[23] F. Piękniewski, "Spontaneous scale-free structures in spike flow graphs for recurrent neural networks," Ph.D. dissertation, Warsaw University, Warsaw, Poland, 2008. [Online]. Available: http://www.mat.umk.pl/~philip/dissertation.pdf

[24] A.-L. Barabási and R. Albert, "Emergence of scaling in random networks," *Science*, no. 286, pp. 509–512, October 1999. [Online]. Available: http://www.sciencemag.org/cgi/content/abstract/286/5439/509

[25] J. Hopfield, "Neural networks and physical systems with emergent collective computational abilities," in *Proceedings of the National Academy of Sciences*. Washington : The Academy, 1982, vol. 79, pp. 2554–2558.

[26] E. Aarts and J. Korst, *Simulated annealing and Boltzmann machines: a stochastic approach to combinatorial optimization and neural computing*. New York, NY, USA: John Wiley & Sons, Inc., 1989.

[27] T. Schreiber, "Spectra of winner-take-all stochastic neural networks," 2008. [Online]. Available: http://www.citebase.org/abstract?id=oai:arXiv.org:0810.3193

[28] F. Piękniewski, "Emergence of scale-free graphs in dynamical spiking neural networks," in *Proc. IEEE International Joint Conference on Neural Networks*. Orlando, Florida, USA: IEEE Press, August 2007, pp. 755–759. [Online]. Available: http://www.ieeexplore.ieee.org/xpl/freeabs_all.jsp?isnumber=4370891&arnumber=4371052&count=569&index=160

[29] E. M. Izhikevich, "Simple model of spiking neurons," *IEEE Transactions on Neural Networks*, no. 14, pp. 1569– 1572, 2003. [Online]. Available: http://www.nsi.edu/users/izhikevich/publications/spikes.pdf

[30] E. M. Izhikevich, J. A. Gally, and G. M. Edelman, "Spike-timing dynamics of neuronal groups," *Cerebral Cortex*, no. 14, pp. 933–944, 2004. [Online]. Available: http://vesicle.nsi.edu/users/izhikevich/publications/reentry.pdf

[31] P. Erdős and A. Rényi, "On random graphs. i." *Publicationes Mathematicae*, vol. 6, pp. 290–297, 1959.

[32] ——, "The evolution of random graphs," *Magyar Tud. Akad. Mat. Kutat Int. Kzl.*, vol. 5, pp. 17–61, 1960.

[33] F. Chung and L. Lu, *Complex Graphs and Networks (Cbms Regional Conference Series in Mathematics)*. Boston, MA, USA: American Mathematical Society, 2006.